Relative motion from known correspondences

Absolute orientation on SE(2) and SE(3) manifolds and its closed-form solution

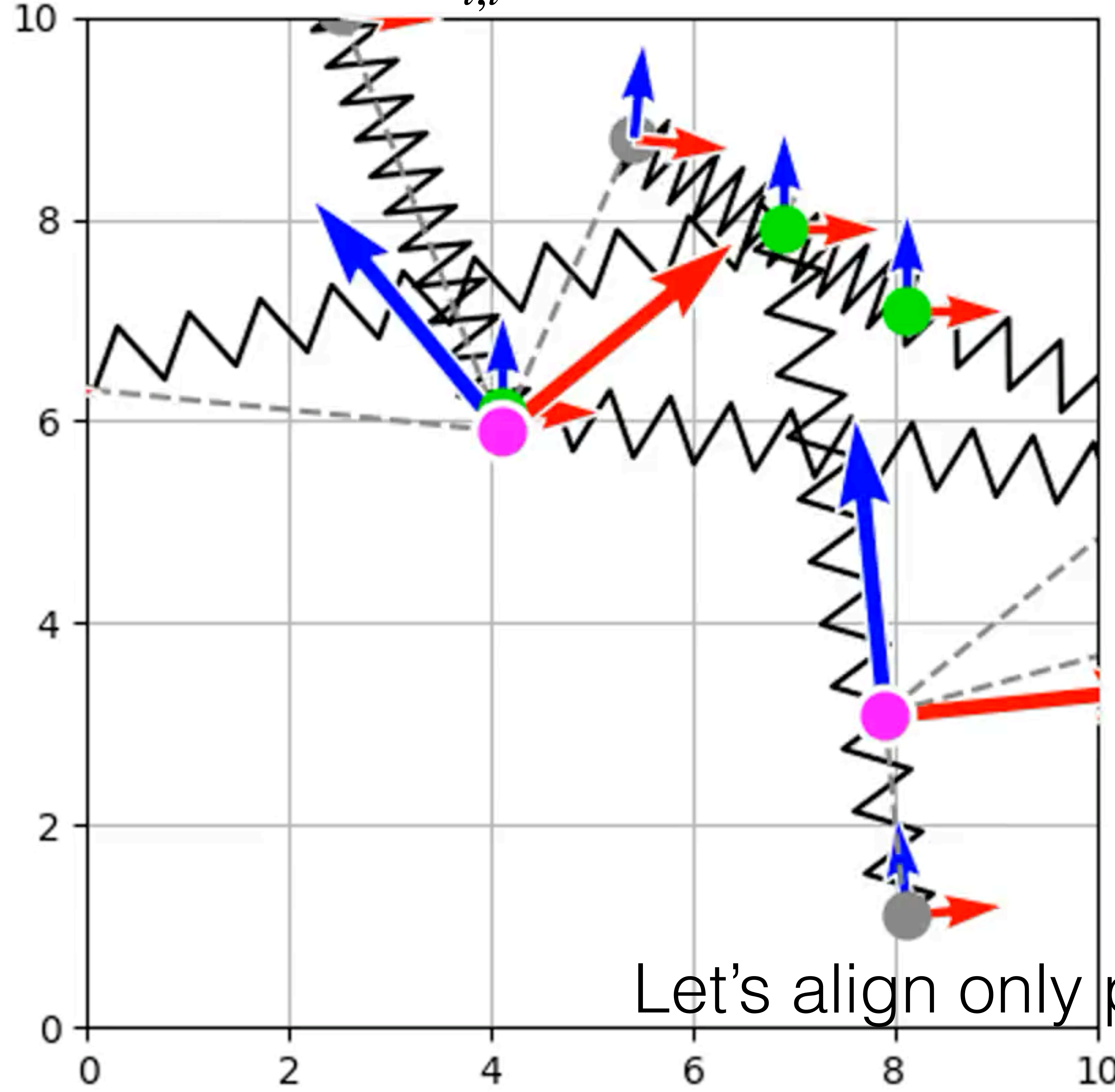
Karel Zimmermann

SLAM from  3D marker detector (RGBD camera)

How many DOF restricted? What is their meaning? Meaning = relative motion

$$\mathbf{x}^\star = \arg\min_{\mathbf{m}_i,\mathbf{x}_t} \sum_{i,t} \|\text{w2r}(\mathbf{m}_i, \mathbf{x}_t) - \mathbf{z}_t^{\mathbf{m}_i}\|^2$$

Do I really need all this complicated stuff?



- $\mathbf{x}_t$ … robot poses
- $\mathbf{m}_i$ … known marker positions
- $\mathbf{z}_t^{\mathbf{m}_i}$ … 3D marker measurements
- …… local coordinate frame
- $\sum_{i,t} \|\text{w2r}(\mathbf{m}_i, \mathbf{x}_t) - \mathbf{z}_t^{\mathbf{m}_i}\|^2$ … marker loss

Let's align only positions (without angle measurements)

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$\mathbf{x}^\star = \arg\min_{\mathbf{m}_i, \mathbf{x}_t}$ "complictaed stuff"

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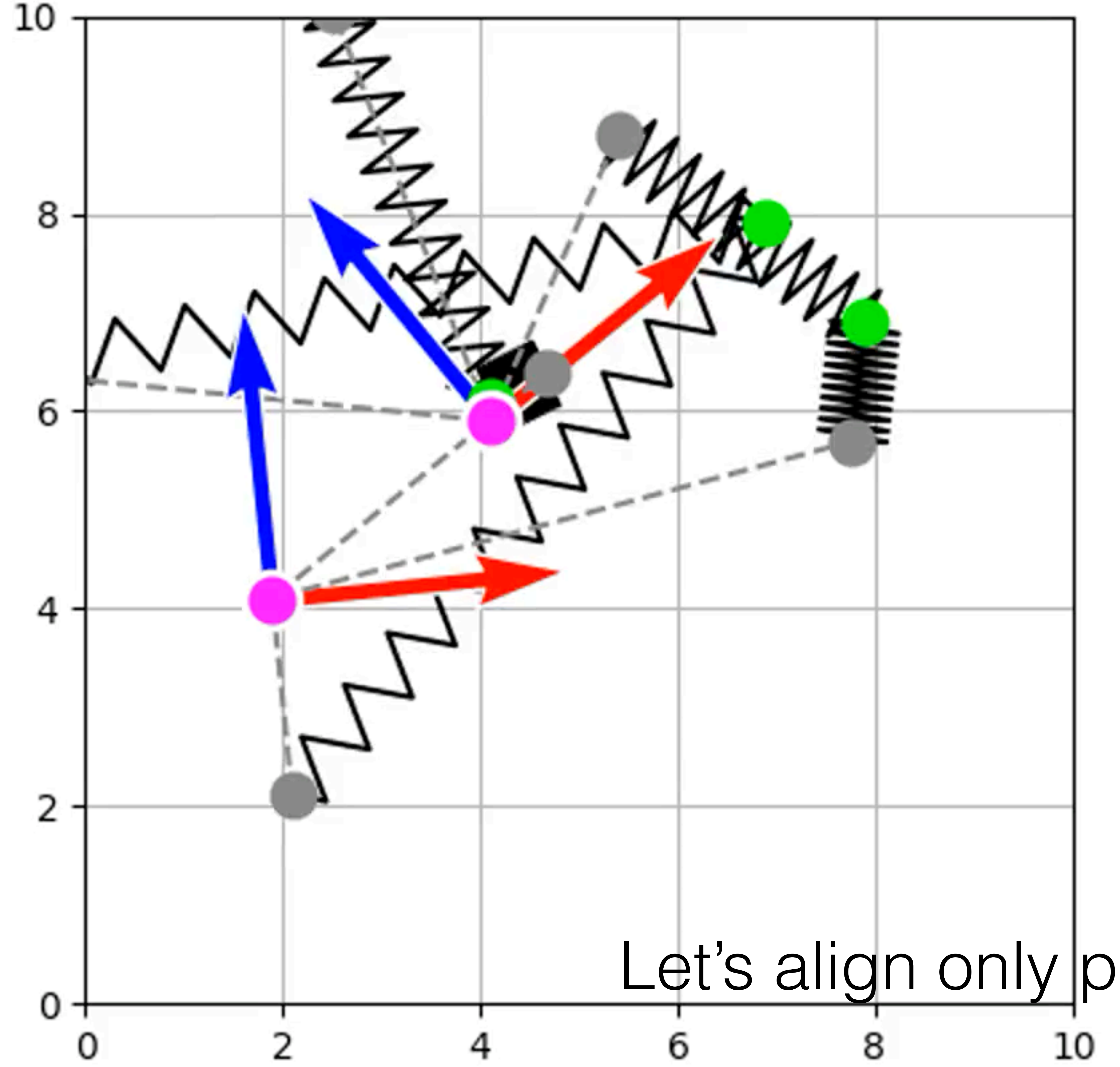

$\mathbf{x}_t$ … robot poses

$\mathbf{m}_i$ … known marker positions

$\boxed{\mathbf{z}_t^{\mathbf{m}_i} \text{ … 2D}}$ marker measurements

…… local coordinate frame

$\sum_{i,t} \|\mathrm{w2r}(\mathbf{m}_i, \mathbf{x}_t) - \mathbf{z}_t^{\mathbf{m}_i}\|^2$ … marker loss

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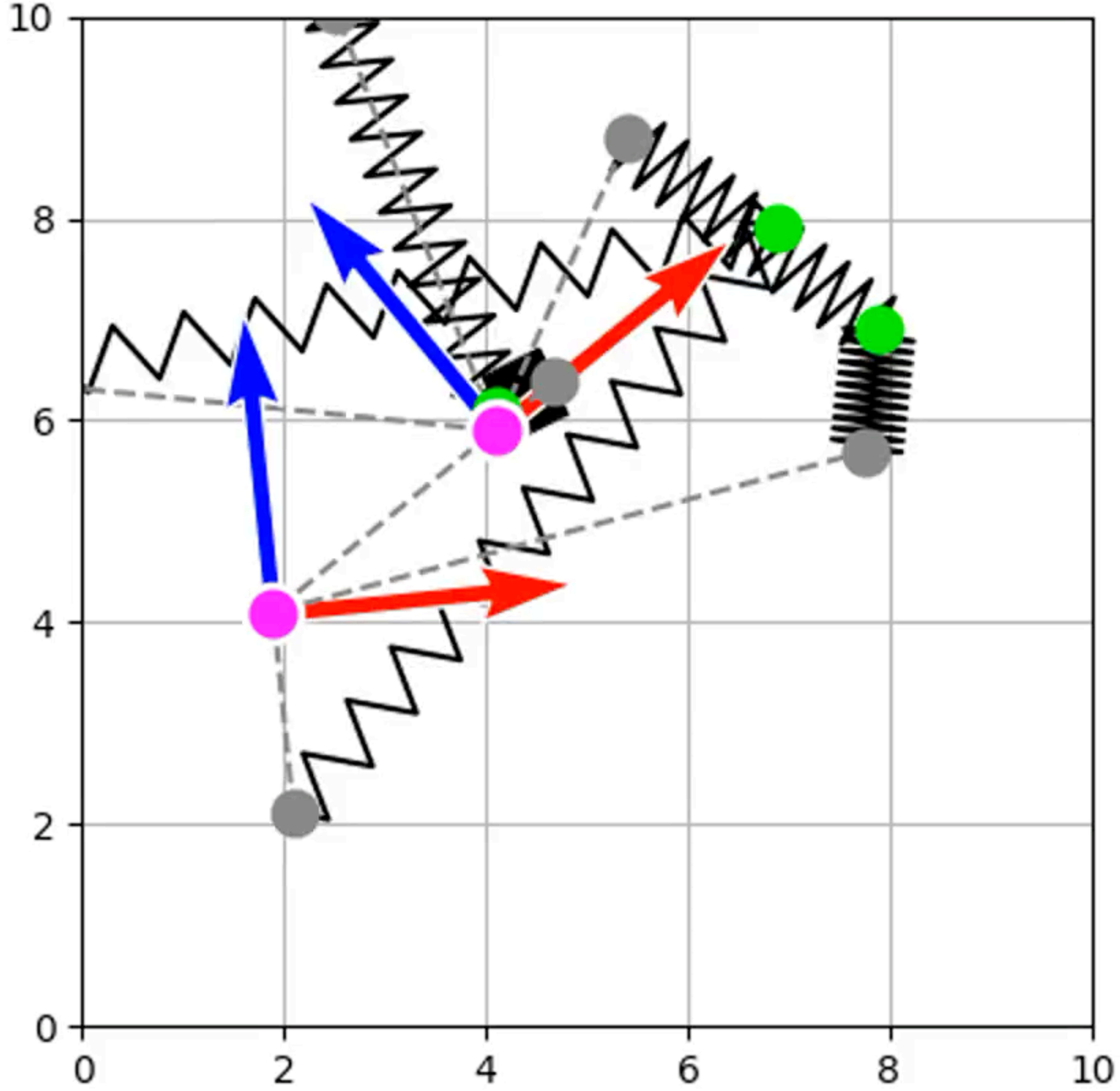

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$\sum_{i,t} \|\mathrm{w2r}(\mathbf{m}_i, \mathbf{x}_t) - \mathbf{z}_t^{\mathbf{m}_i}\|^2$ … marker loss

Let’s fix $\mathbf{x}_1$

SLAM from  3D marker detector (RGBD camera)

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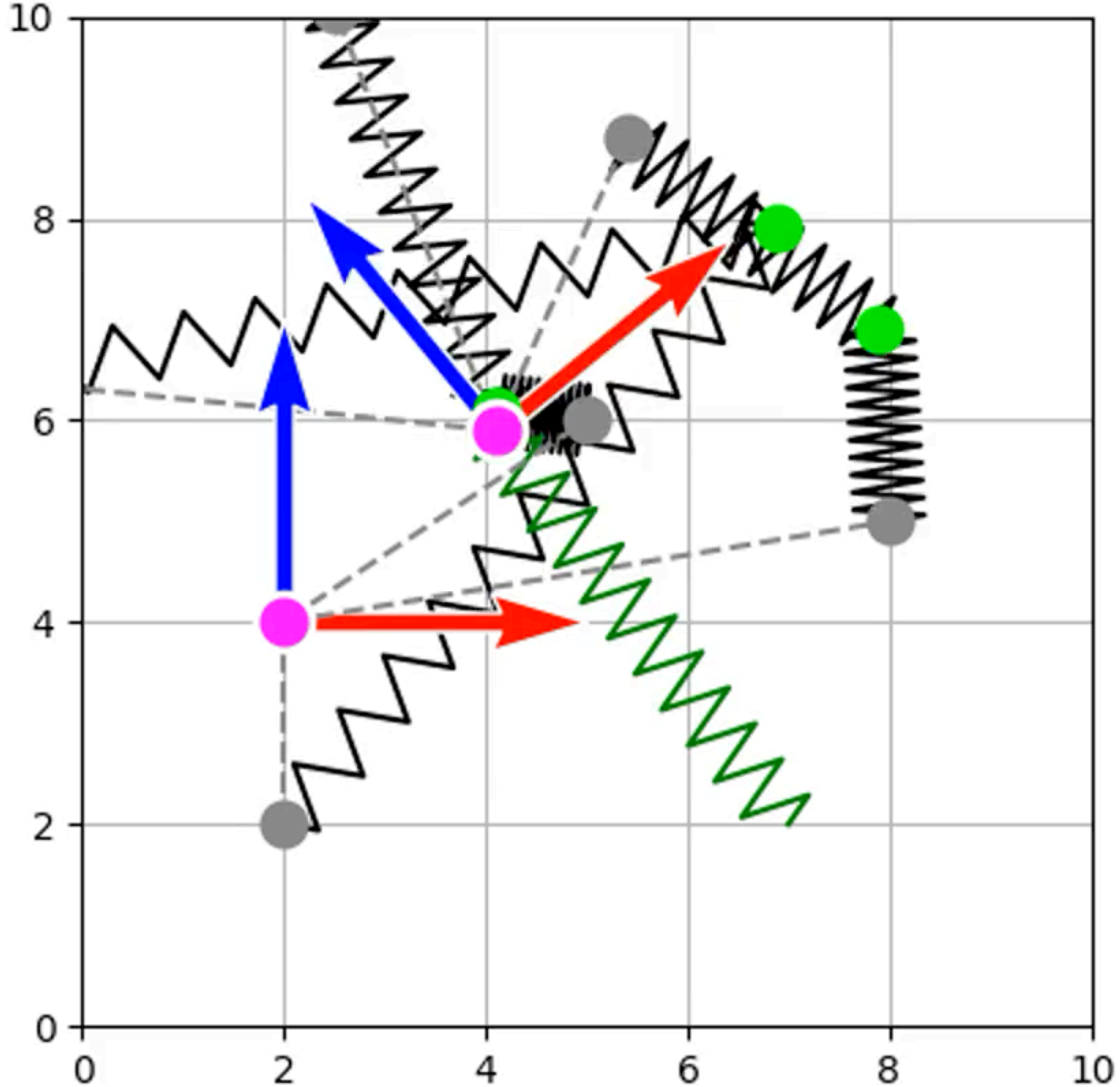

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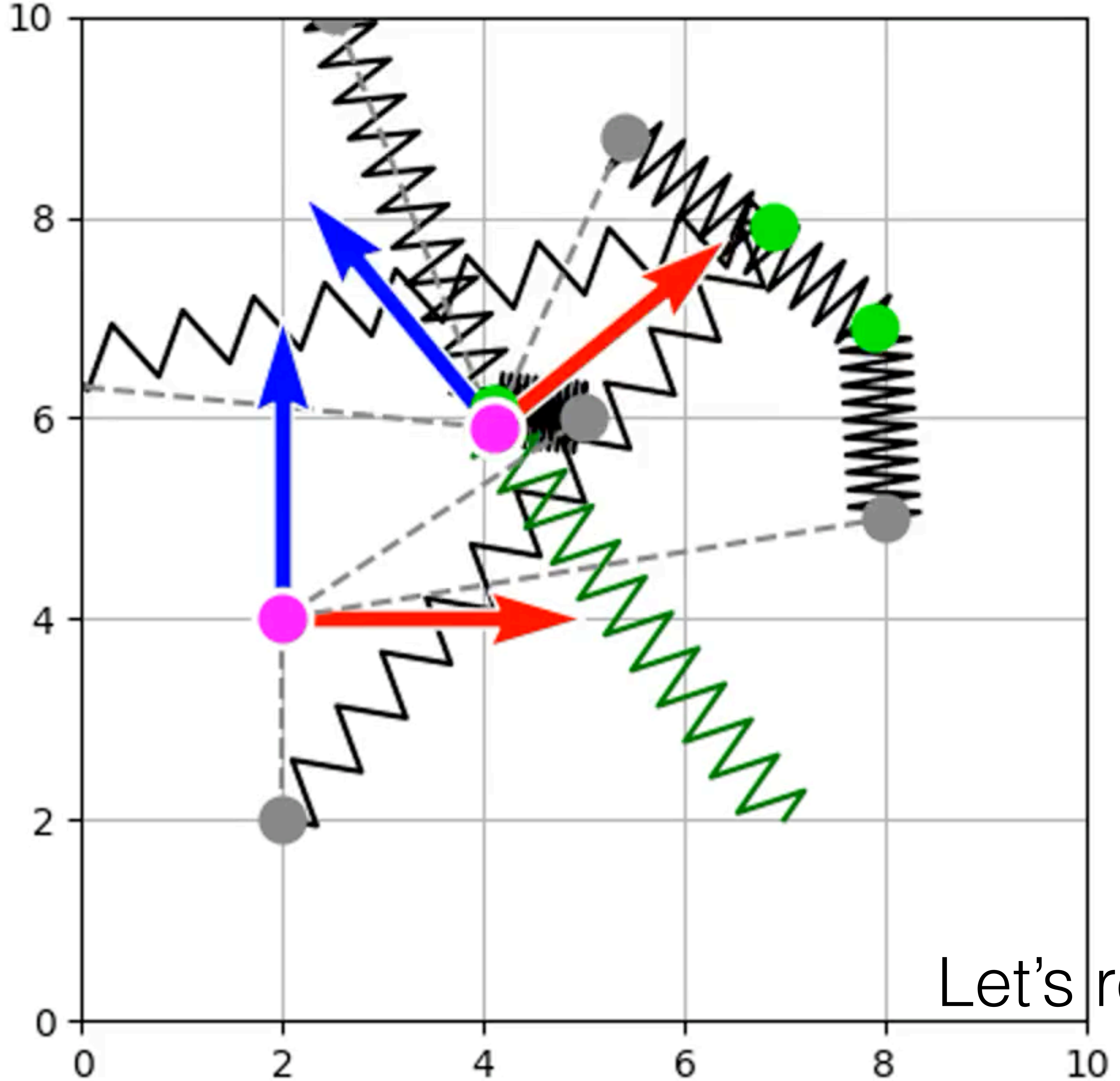


$\mathbf{x}_t$ ... robot poses

$\boxed{\mathbf{m}_i}$ ... known marker positions

$\mathbf{z}_t^{\mathbf{m}_i}$ ... 2D marker measurements

...... local coordinate frame

$\sum_{i,t} \|\mathrm{w2r}(\mathbf{m}_i, \mathbf{x}_t) - \mathbf{z}_t^{\mathbf{m}_i}\|^2$ ... marker loss

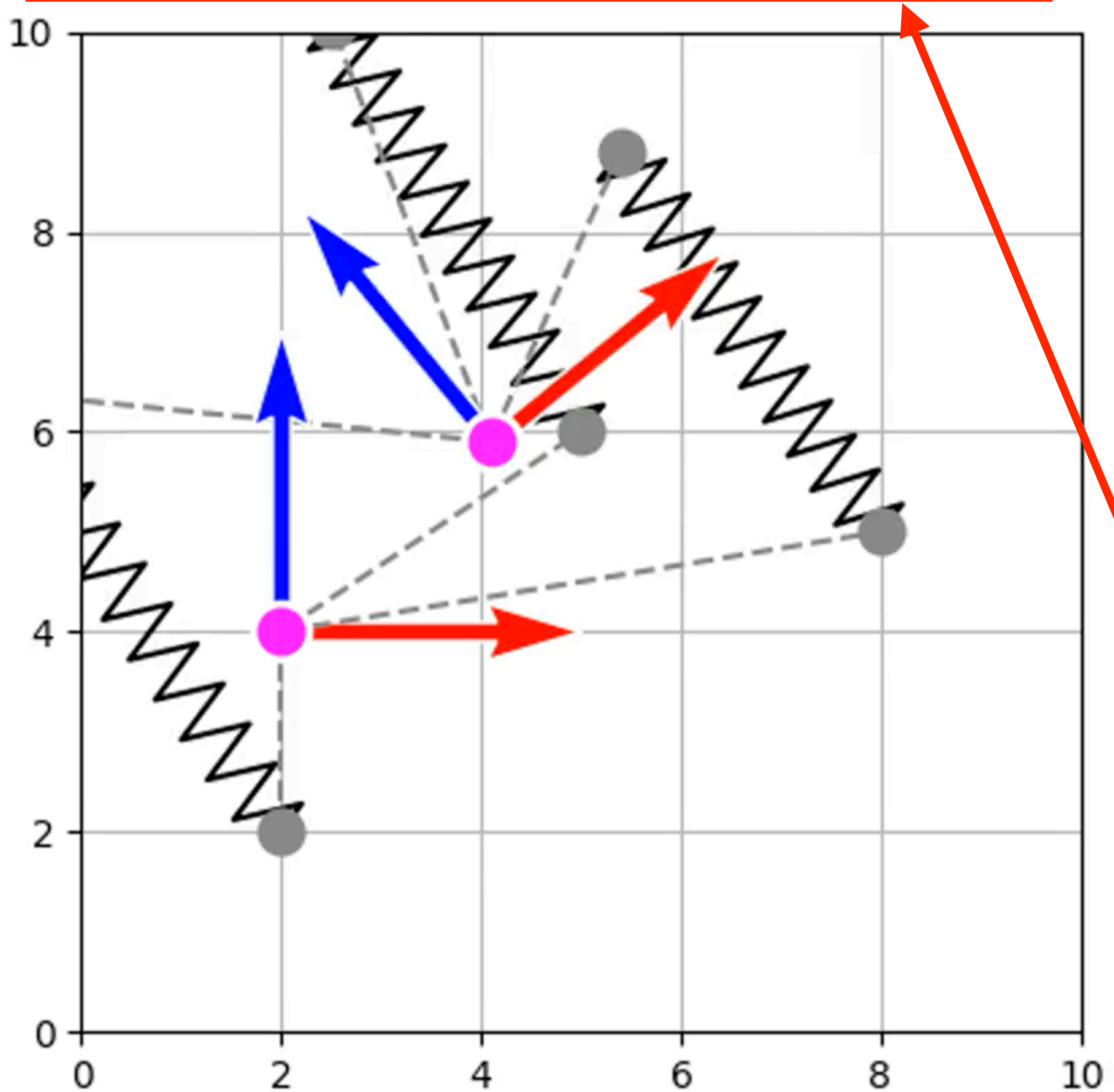
Let's remove markers and attract pcl to each other

SLAM from  3D marker detector (RGBD camera)

How many DOF restricted? What is their meaning? Meaning = relative motion

$$\boxed{\mathbf{x}^\star = \arg\min_{\mathbf{m}_i, \mathbf{x}_t} \text{"complicted stuff"}}$$

Do I really need all this complicated stuff?



$\mathbf{x}_t$ … robot poses

$\mathbf{z}_t^{\mathbf{m}_i}$ … 2D marker measurements

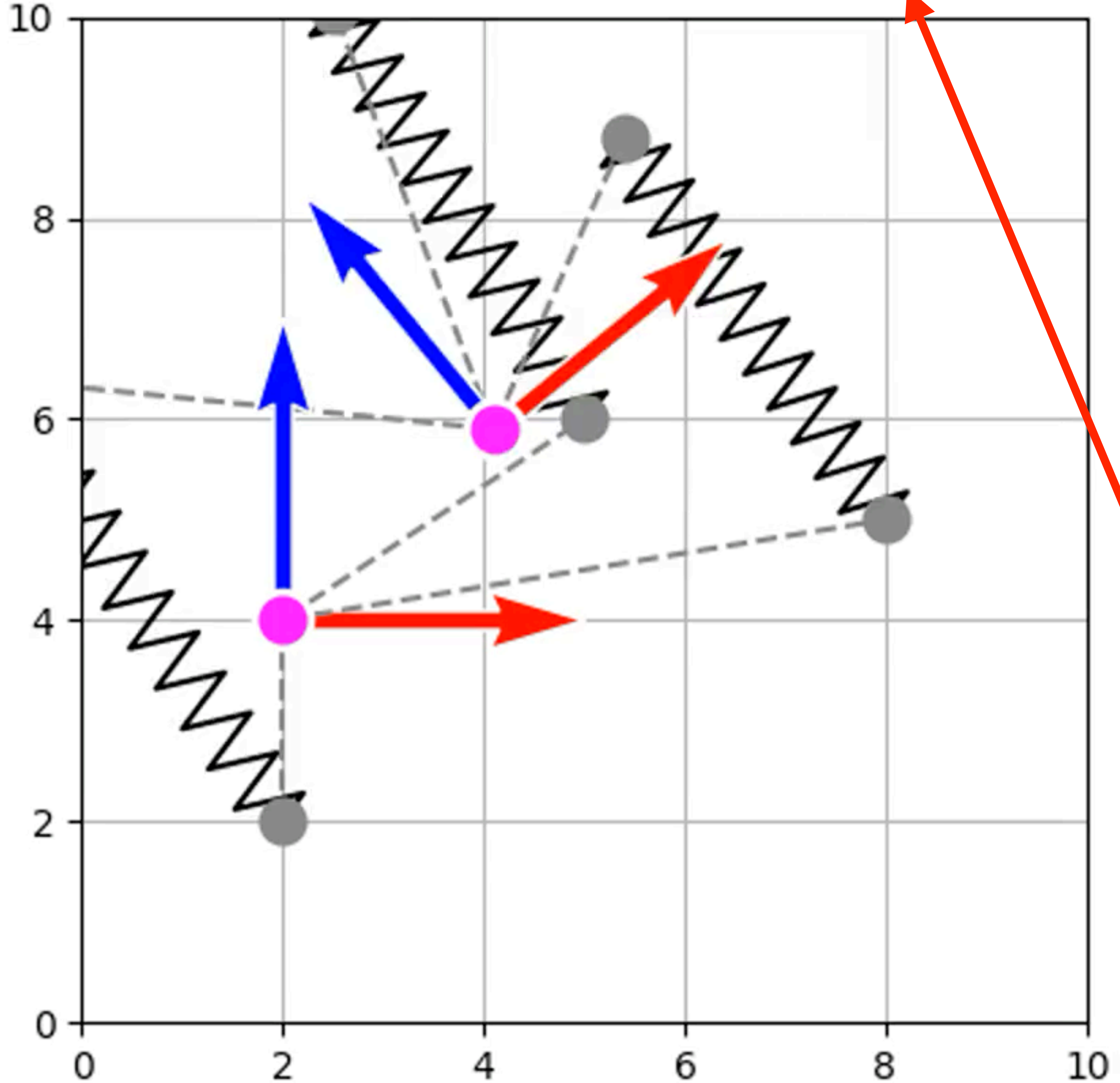
…… local coordinate frame

$\sum_{i,t} \|\text{w2r}(\mathbf{m}_i, \mathbf{x}_t) - \mathbf{z}_t^{\mathbf{m}_i}\|^2$ … marker loss

Let's simplify the "complicated stuff"

SLAM from  3D marker detector (RGBD camera)

$$\mathbf{z}^{\mathrm{odom}} = \arg\min_{x,y,\theta} \sum_i \left\| \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \mathbf{z}_1^{\mathbf{m}_i} + \begin{bmatrix} x \\ y \end{bmatrix} - \mathbf{z}_2^{\mathbf{m}_i} \right\|^2 = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{z}_1^{\mathbf{m}_i} + \mathbf{t} - \mathbf{z}_2^{\mathbf{m}_i} \right\|^2$$



$\mathbf{x}_t$ … robot poses

$\mathbf{z}_t^{\mathbf{m}_i}$ … 2D marker measurements

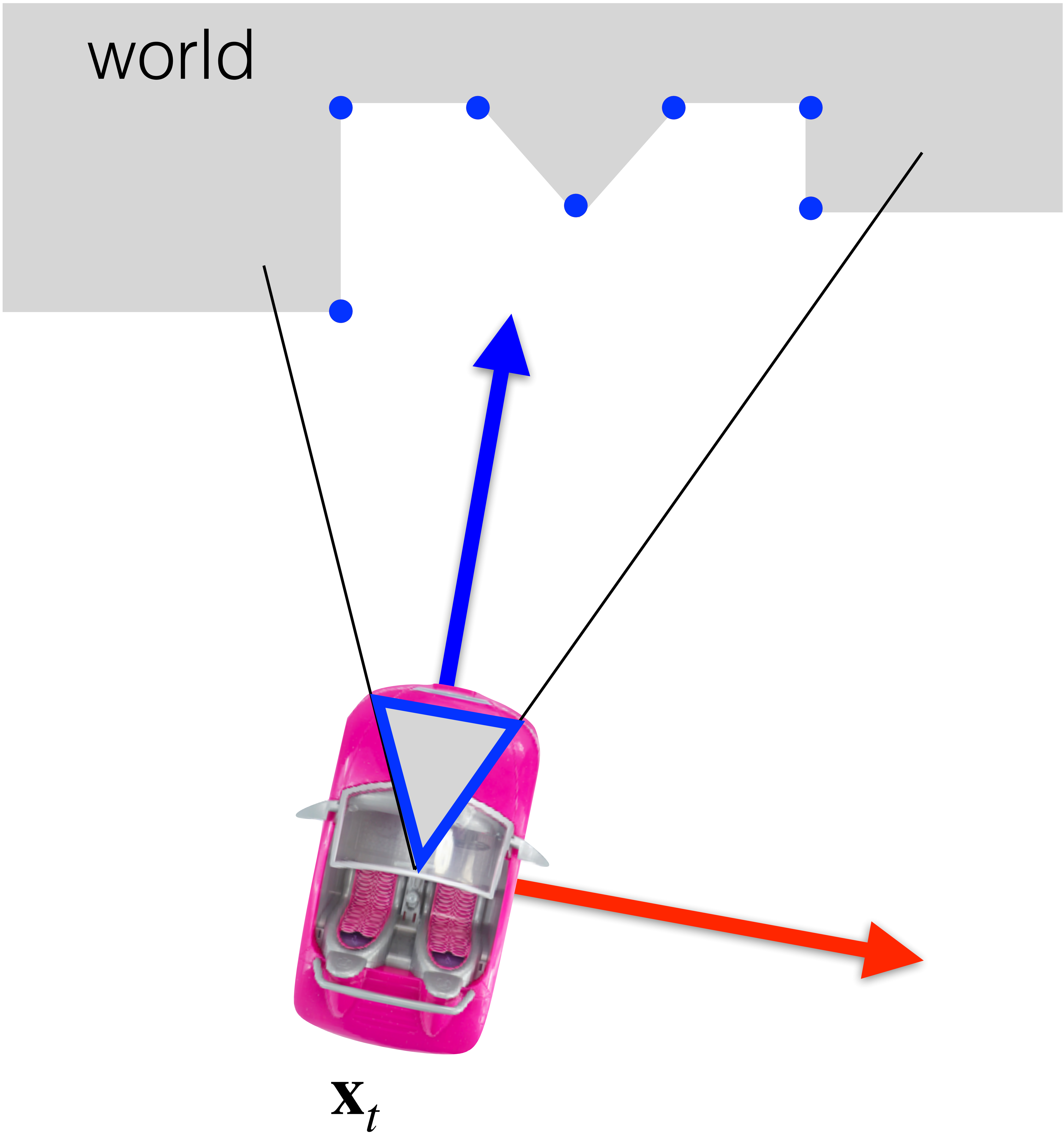
…… local coordinate frame

$\sum_{i,t} \|\mathrm{w2r}(\mathbf{m}_i, \mathbf{x}_t) - \mathbf{z}_t^{\mathbf{m}_i}\|^2$ … marker loss

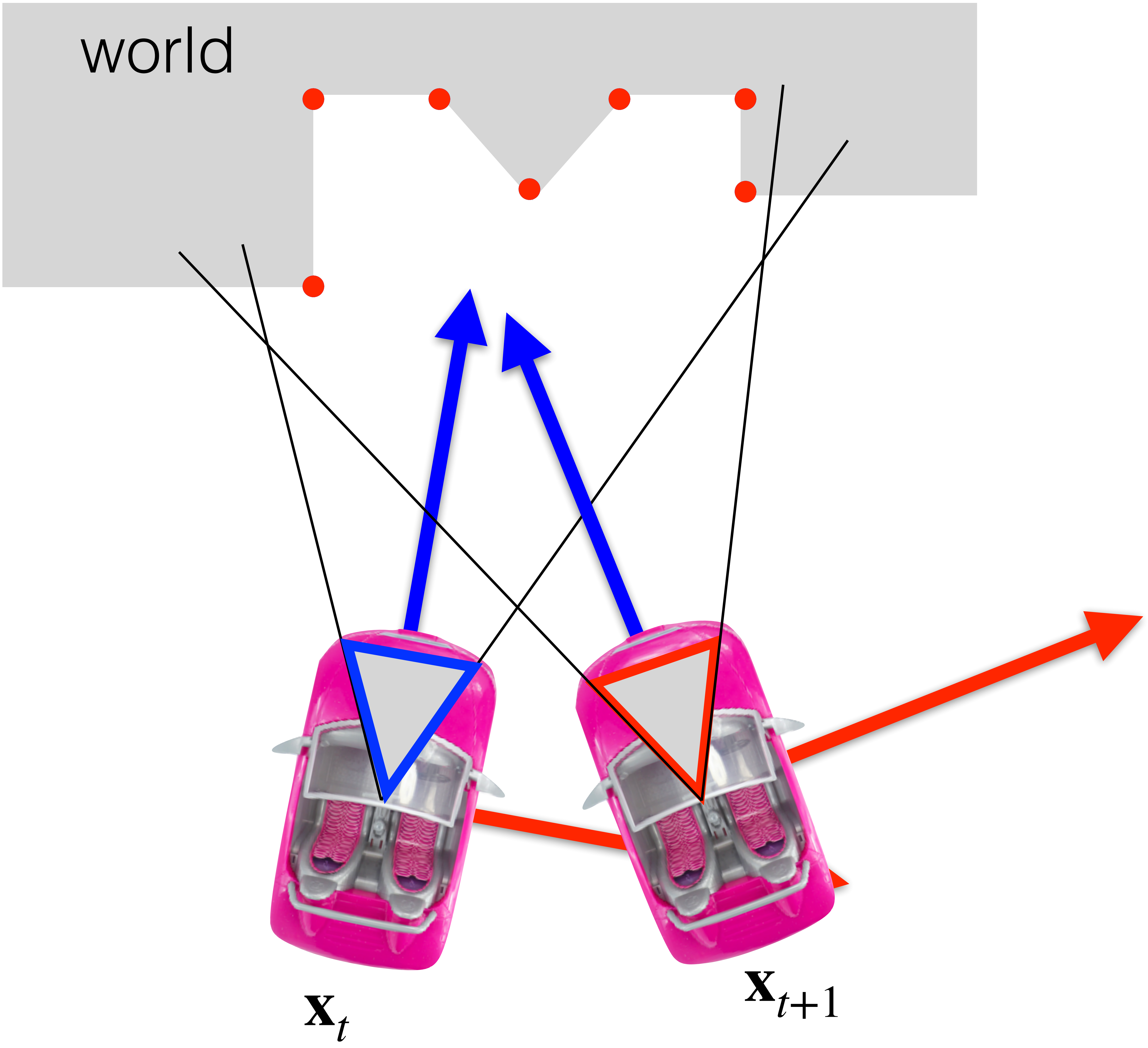
Is it really significant improvement?

Let's simplify the "complicated stuff"

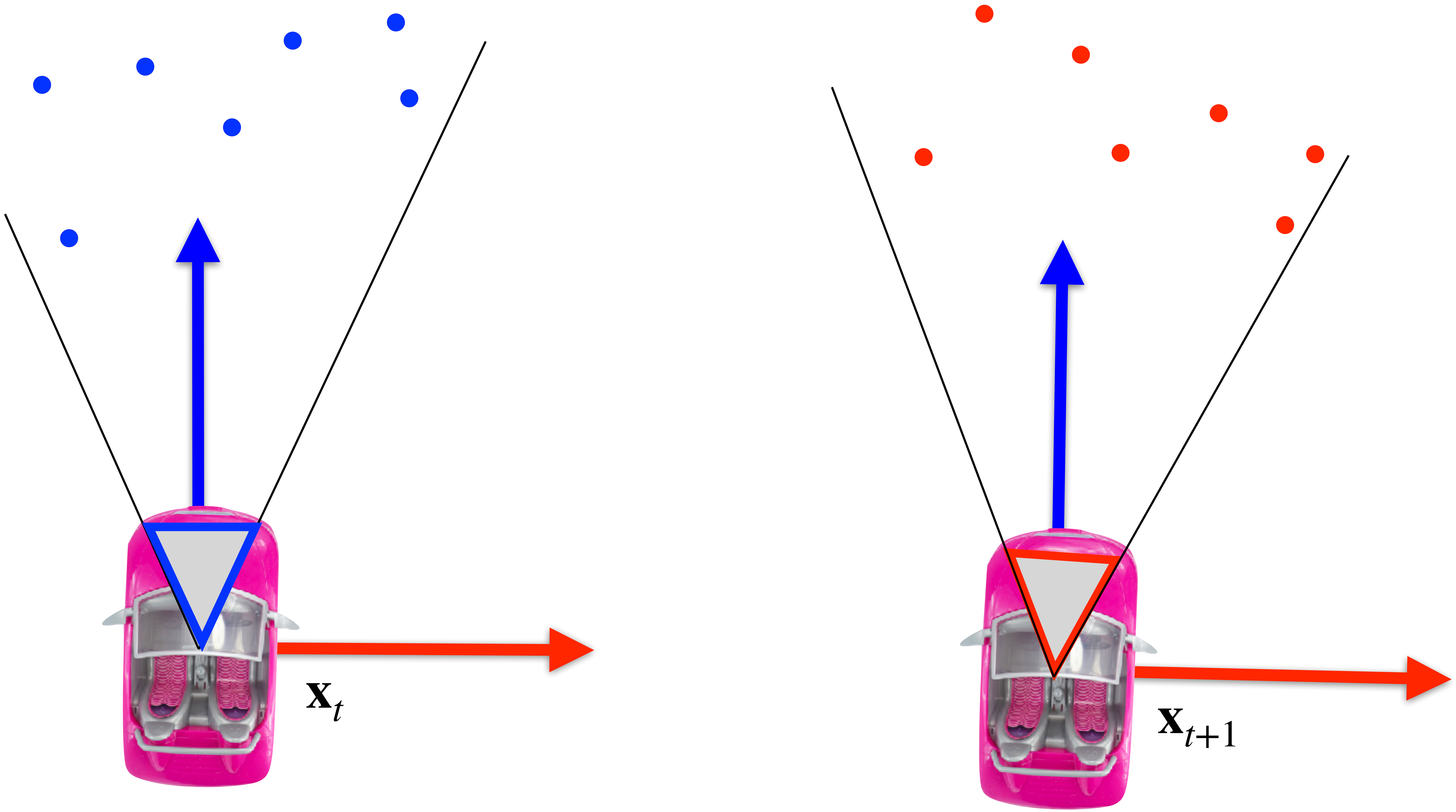
Pose estimation from known correspondences



Pose estimation from known correspondences

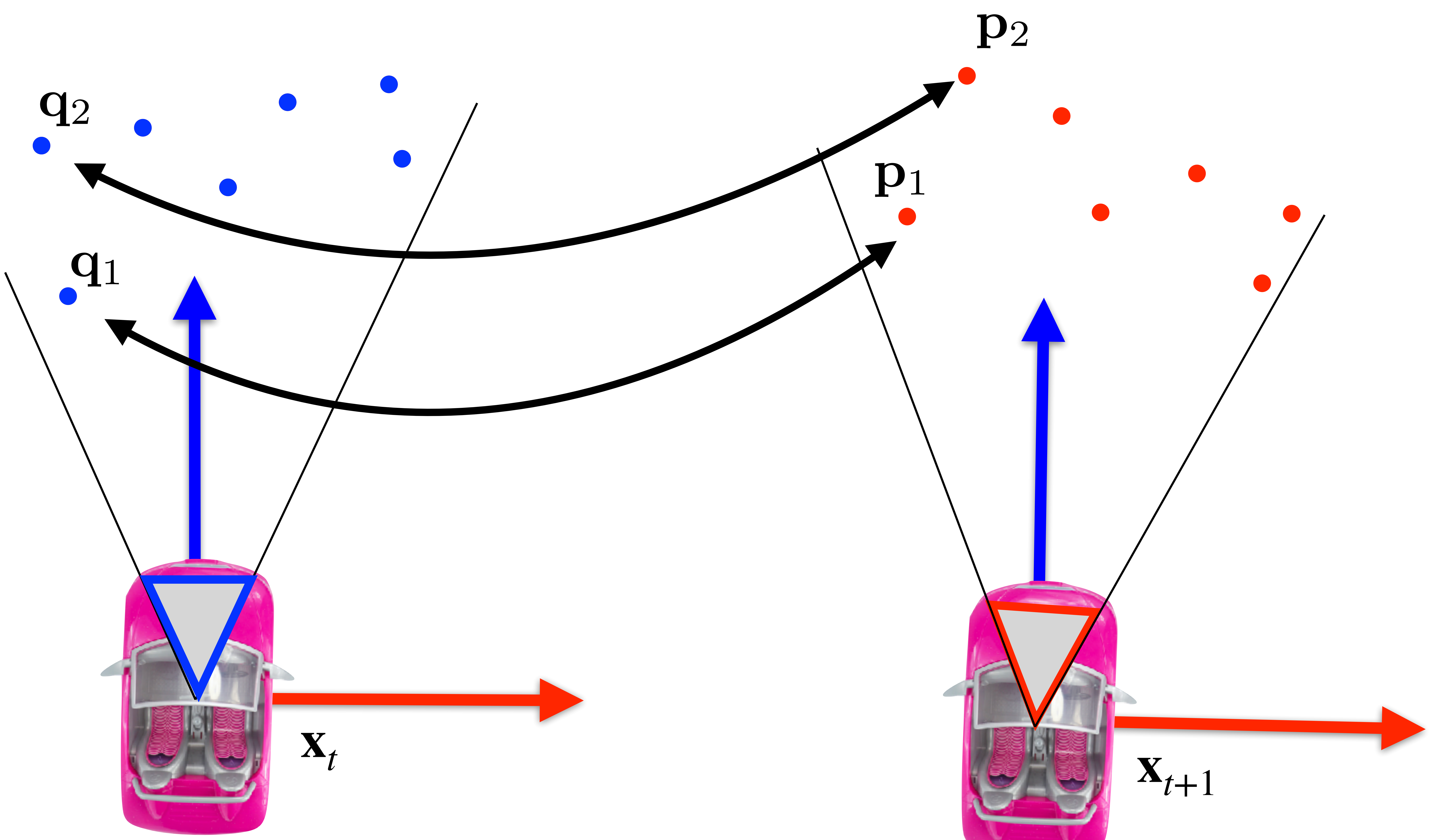


Pose estimation from known correspondences



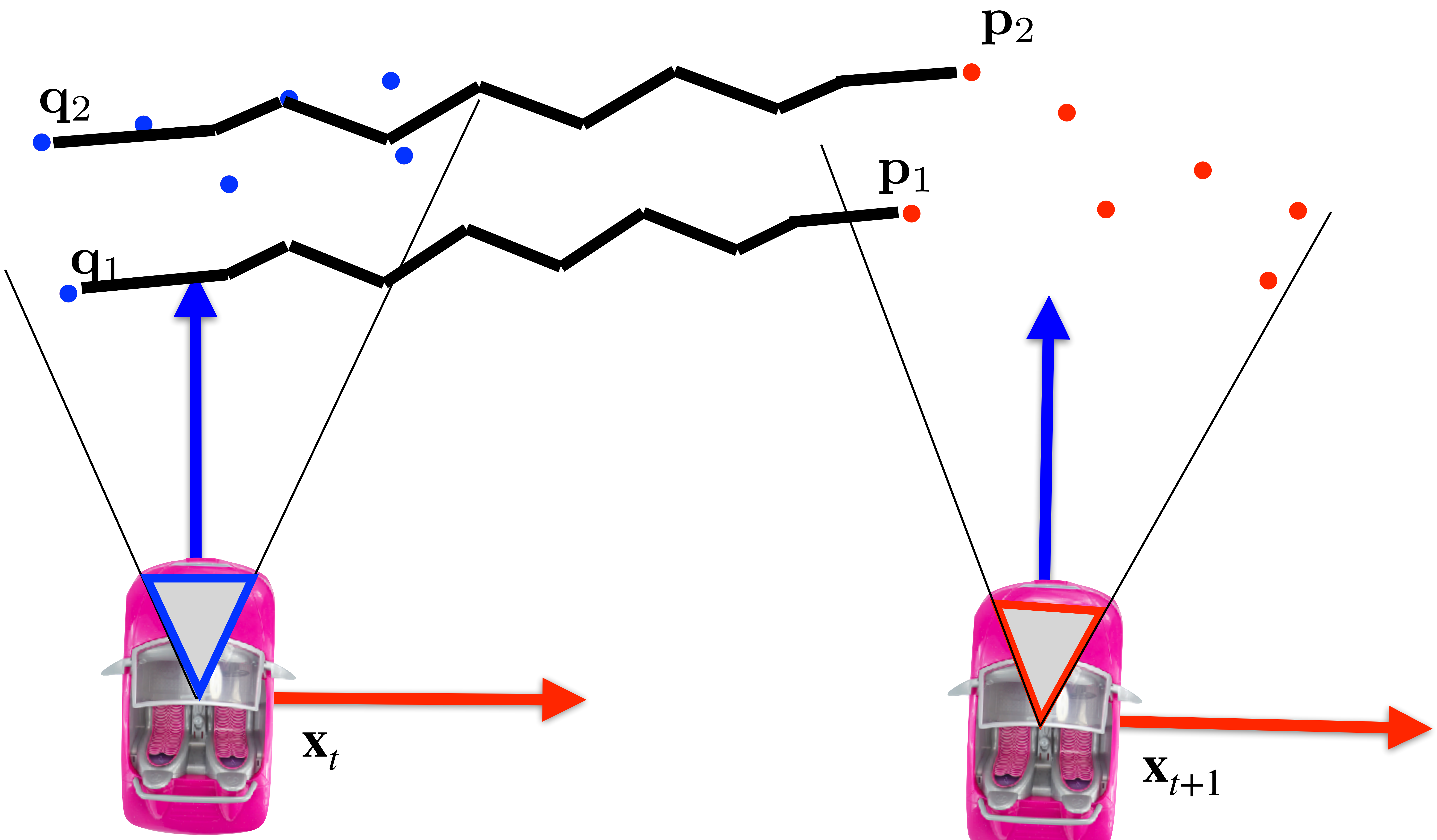
Pose estimation from known correspondences

Assume 2D-2D correspondences known



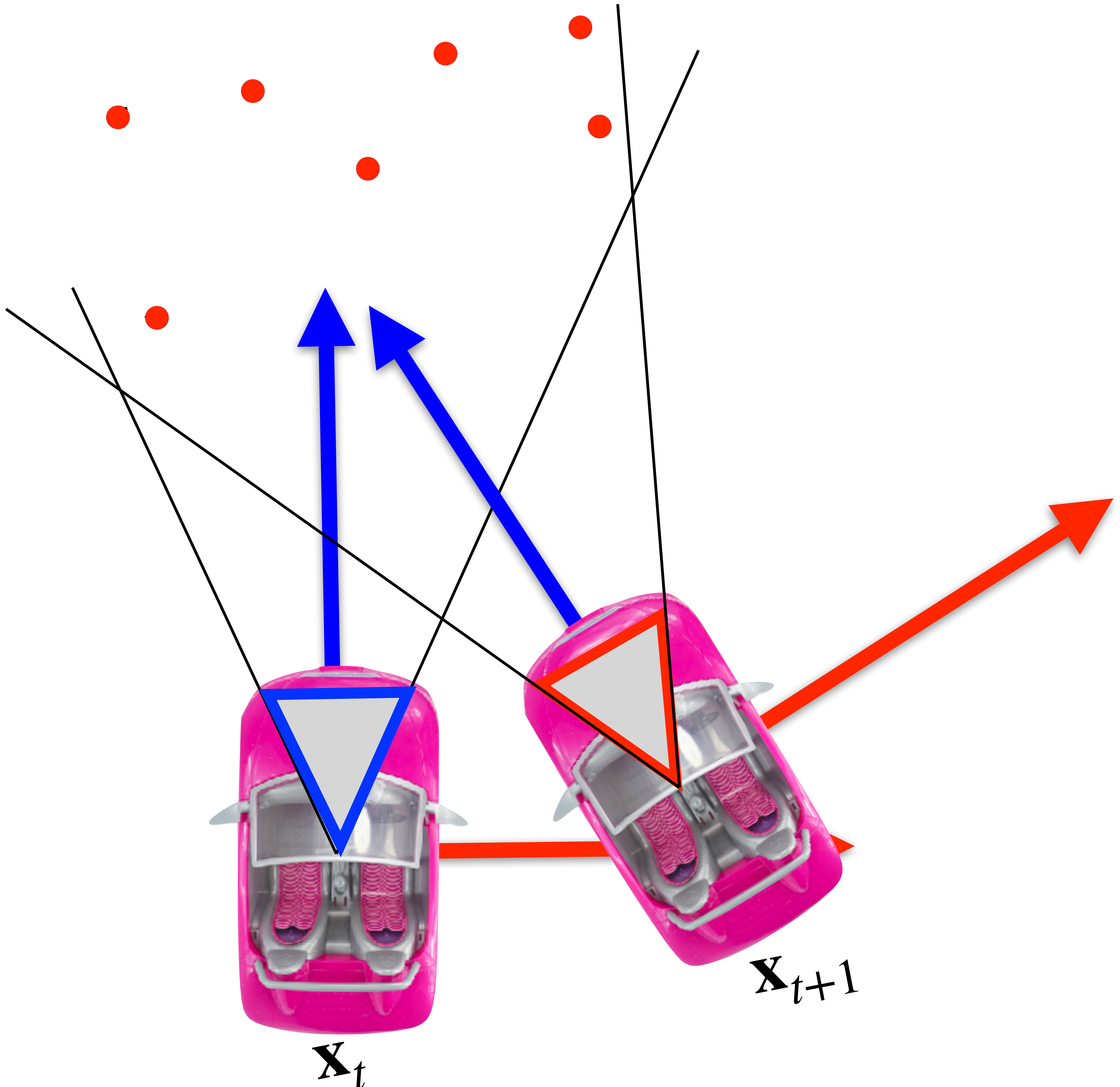
Pose estimation from known correspondences

Estimate odometry measurement: $\mathbf{z}^{\text{odom}} = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \right\|^2$



Pose estimation from known correspondences

Estimate odometry measurement: $\mathbf{z}^{\text{odom}} = \arg\min_{\mathbf{t},\theta} \sum_i \| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \|^2$



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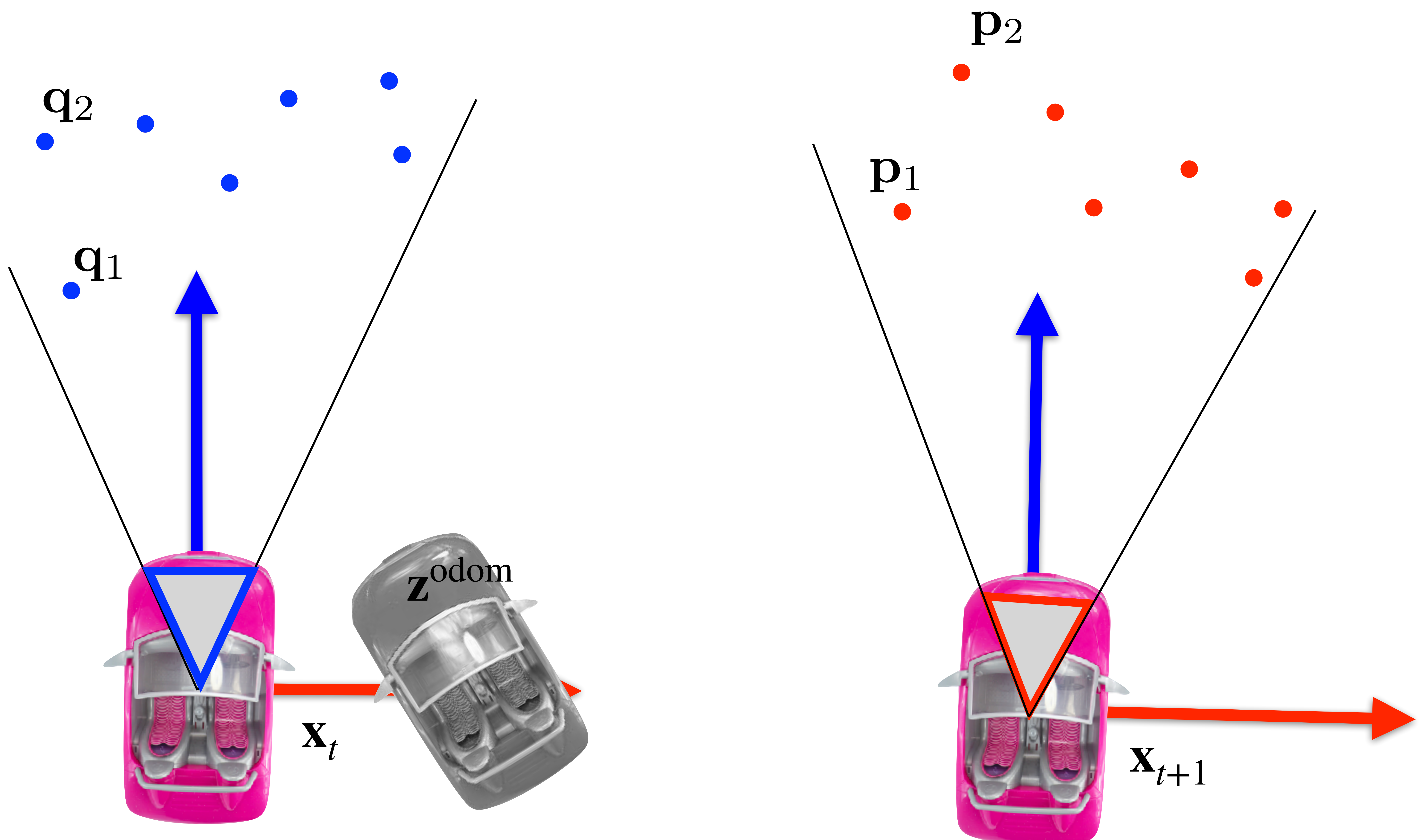
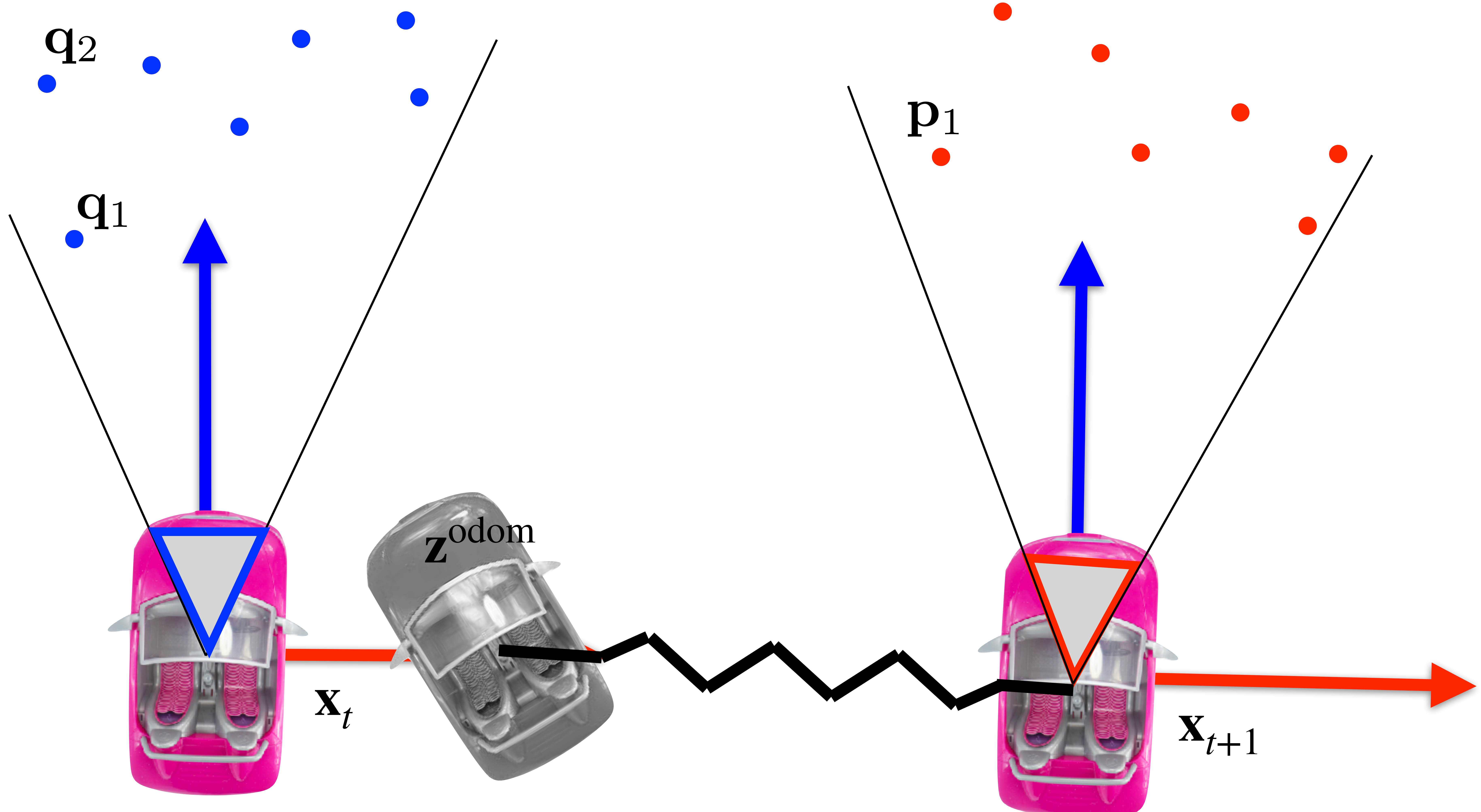


Pose estimation from known correspondences

Estimate odometry measurement: $\mathbf{z}^{\text{odom}} = \arg\min_{\mathbf{t},\theta} \sum_i \| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \|^2$

Apply only single odometry factor: $\arg\min_{\mathbf{x}_t, \mathbf{x}_{t+1}} \|\text{w2r}(\mathbf{x}_{t+1}, \mathbf{x}_t) - \mathbf{z}^{\text{odom}})\|^2$

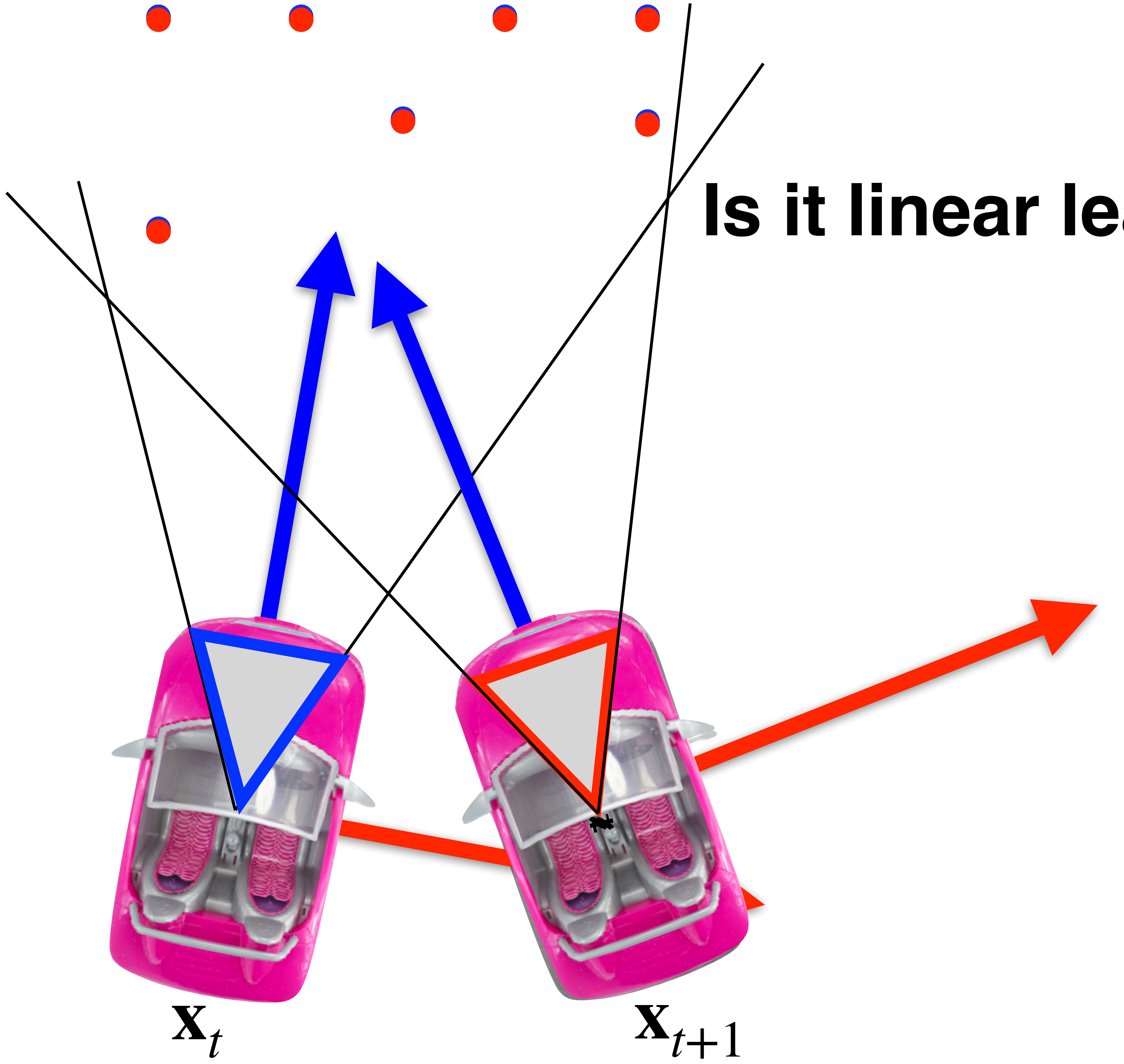


Pose estimation from known correspondences

Estimate odometry measurement: $\mathbf{z}^{\text{odom}} = \arg\min_{\mathbf{t},\theta} \sum_i \| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \|^2$

Apply only single odometry factor: $\arg\min_{\mathbf{x}_t, \mathbf{x}_{t+1}} \|\text{w2r}(\mathbf{x}_{t+1}, \mathbf{x}_t) - \mathbf{z}^{\text{odom}})\|^2$

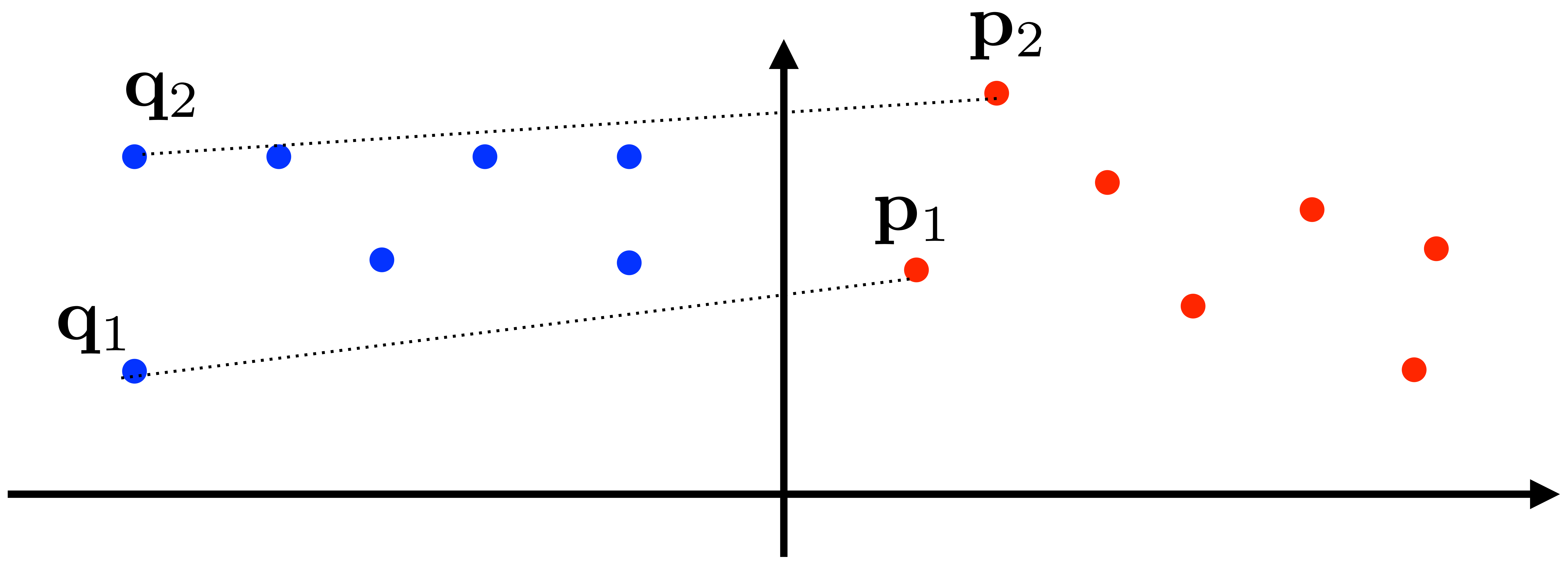
Is it linear least squares problem



Absolute orientation problem in SE(2)

$$\mathbf{z}^{\text{odom}} = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \right\|^2$$

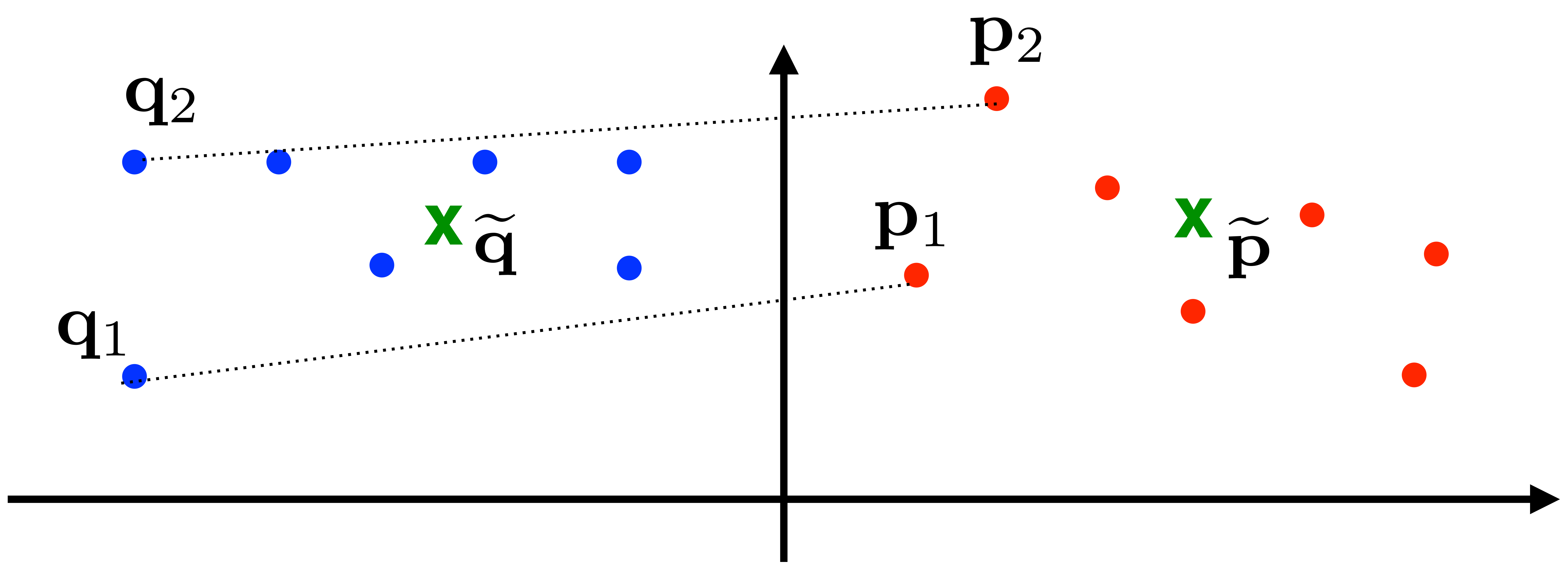
Substitution:



Absolute orientation problem in SE(2)

$$\mathbf{z}^{\text{odom}} = \arg\min_{\mathbf{t},\theta} \sum_i \| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \|^2$$

Substitution: $\mathbf{p}_i' = \mathbf{p}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}, \quad \mathbf{q}_i' = \mathbf{q}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$



Absolute orientation problem in SE(2)

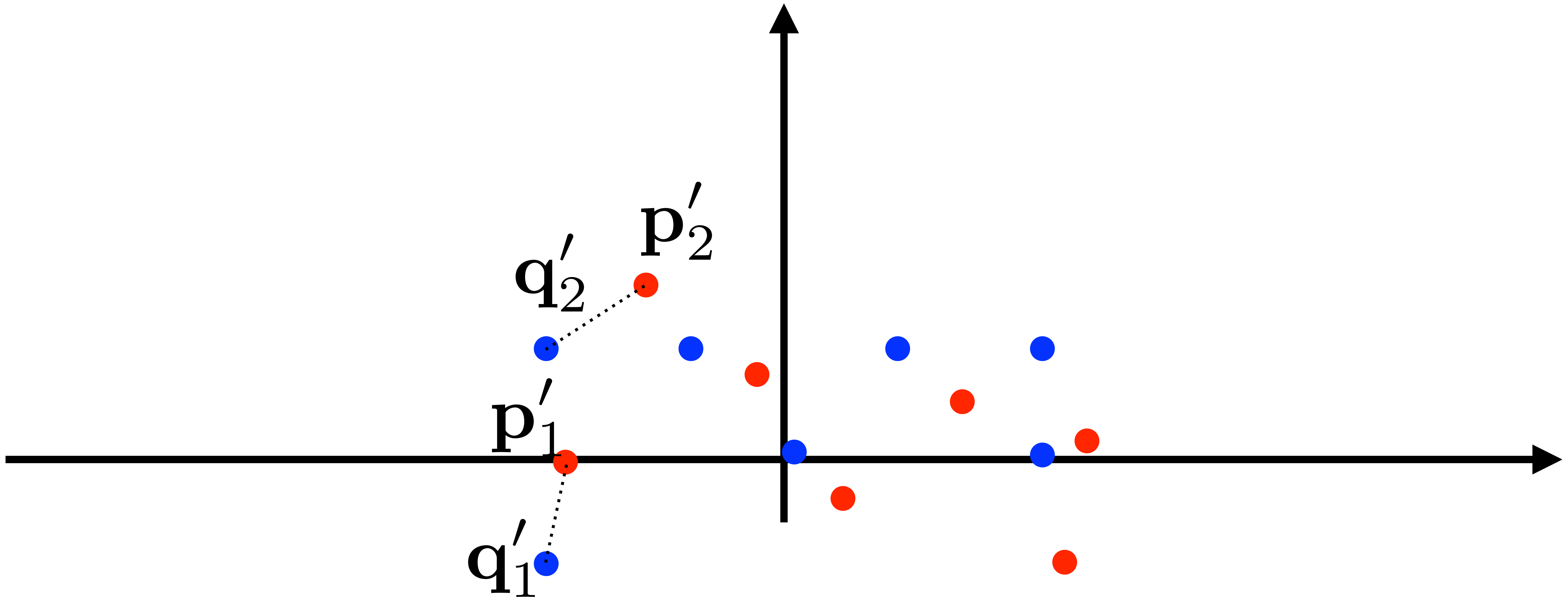
$$\mathbf{z}^{\text{odom}} = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \overset{\mathbf{p}' + \tilde{\mathbf{p}}}{\mathbf{p}_i} + \mathbf{t} - \overset{\mathbf{q}' + \tilde{\mathbf{q}}}{\mathbf{q}_i} \right\|^2 = \arg\min_{\mathbf{t},\theta} \sum_i \|\mathbf{R}_\theta \mathbf{p}_i' - \mathbf{q}_i'\|^2 + \|\mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2$$

Can be always zero by appropriate choice of $\mathbf{t}$

Substitution: $\mathbf{p}_i' = \mathbf{p}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}, \quad \mathbf{q}_i' = \mathbf{q}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

Depends only on θ

Solution: $\theta^\star = \arg\min_\theta \sum_i \|\mathbf{R}_\theta \mathbf{p}_i' - \mathbf{q}_i'\|^2$



Absolute orientation problem in SE(2)

$$\mathbf{z}^{\text{odom}} = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \right\|^2 = \arg\min_{\mathbf{t},\theta} \sum_i \|\mathbf{R}_\theta \mathbf{p}_i' - \mathbf{q}_i'\|^2 + \|\mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2$$

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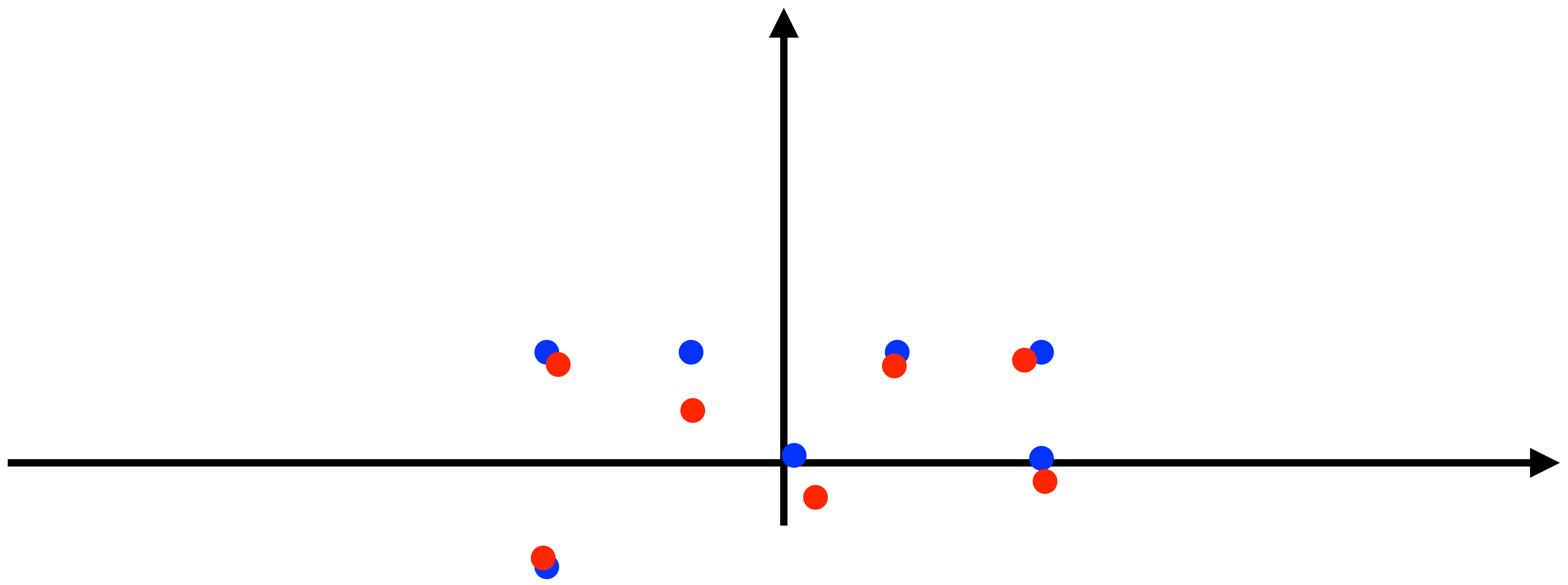

Absolute orientation problem in SE(2)

$$\mathbf{z}^{\text{odom}} = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \right\|^2 = \arg\min_{\mathbf{t},\theta} \sum_i \|\mathbf{R}_\theta \mathbf{p}_i' - \mathbf{q}_i'\|^2 + \|\mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2$$

Substitution: $\mathbf{p}_i' = \mathbf{p}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}, \quad \mathbf{q}_i' = \mathbf{q}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

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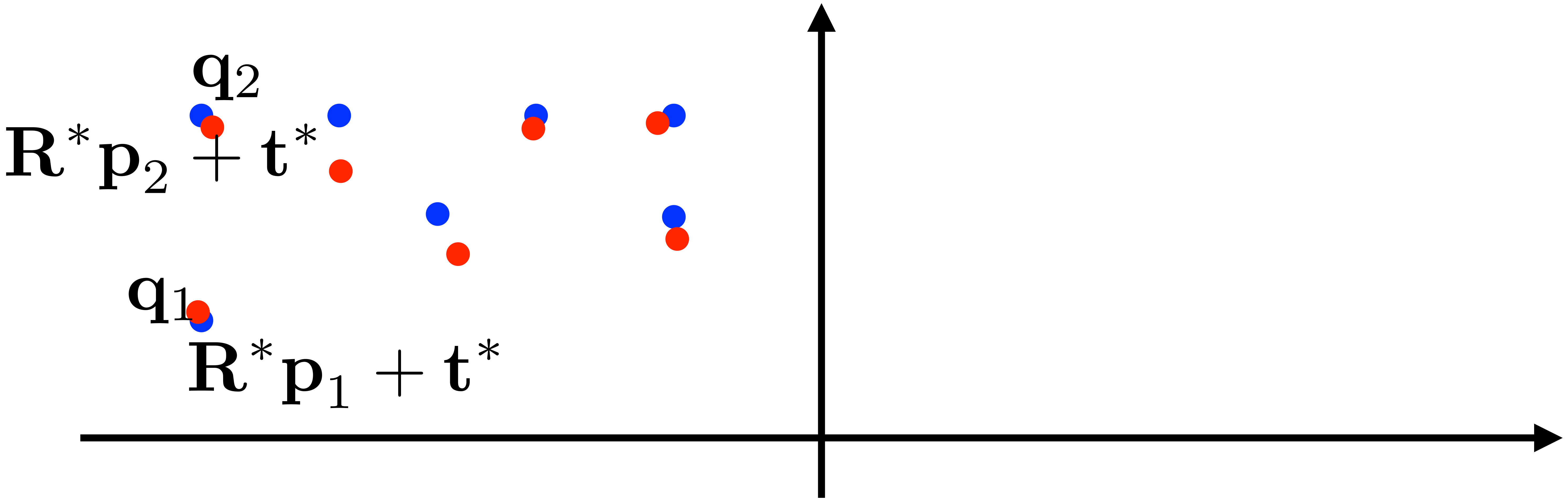


Absolute orientation problem in SE(2)

$$\mathbf{z}^v = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p} + \mathbf{t} - \mathbf{q} \right\|^2 = \arg\min_{\mathbf{t},\theta} \sum_i \|\mathbf{R}_\theta \mathbf{p}_i' - \mathbf{q}_i'\|^2 + \|\mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2$$

Substitution: $\mathbf{p}_i' = \mathbf{p}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}, \quad \mathbf{q}_i' = \mathbf{q}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

Can be always zero
by appropriate choice of $\mathbf{t}$

Solution:

$$\theta^\star = \arg\min_\theta \sum_i \|\mathbf{R}_\theta \mathbf{p}_i' - \mathbf{q}_i'\|^2 = \textbf{???}$$

$$\mathbf{t}^\star = \arg\min_{\mathbf{t}} \|\mathbf{R}_{\theta^\star} \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2 = \tilde{\mathbf{q}} - \mathbf{R}_{\theta^\star}\tilde{\mathbf{p}}$$

Absolute orientation problem in SE(2)

$$\theta^\star = \arg\min_\theta \sum_i \|\mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i\|^2$$

$$= \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left\| \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} - \begin{bmatrix} q'_x \\ q'_y \end{bmatrix} \right\|^2$$

$$= \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left\| \begin{matrix} \cos(\theta)p'_x - \sin(\theta)p'_y - q'_x \\ \sin(\theta)p'_x + \cos(\theta)p'_y - q'_y \end{matrix} \right\|^2$$

Absolute orientation problem in SE(2)

$$\theta^\star = \arg\min_\theta \sum_i \|\mathbf{R}_\theta \mathbf{p}_i' - \mathbf{q}_i'\|^2$$

$$= \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left\| \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} p_x' \\ p_y' \end{bmatrix} - \begin{bmatrix} q_x' \\ q_y' \end{bmatrix} \right\|^2$$

$$= \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left\| \begin{matrix} \cos(\theta)p_x' - \sin(\theta)p_y' - q_x' \\ \sin(\theta)p_x' + \cos(\theta)p_y' - q_y' \end{matrix} \right\|^2$$

$$= \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left(p_x' \cos(\theta) - p_y' \sin(\theta) - q_x'\right)^2 + \left(p_x' \sin(\theta) + p_y' \cos(\theta) - q_y'\right)^2$$

Absolute orientation problem in SE(2)

$$\theta^\star = \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left(p'_x\cos(\theta) - p'_y\sin(\theta) - q'_x\right)^2 + \left(p'_x\sin(\theta) + p'_y\cos(\theta) - q'_y\right)^2$$

Derivative: $$\sum_{\mathbf{p}',\mathbf{q}'} 2(p'_x\cos(\theta) - p'_y\sin(\theta) - q'_x)\cdot(-p'_x\sin(\theta) - p'_y\cos(\theta))$$
$$+ \quad 2(p'_x\sin(\theta) + p'_y\cos(\theta) - q'_y)\cdot(p'_x\cos(\theta) - p'_y\sin(\theta)) = 0$$

Simplify: $$\sum_{\mathbf{p}',\mathbf{q}'} p'^2_x \cdot \left(-\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta)\right)$$
$$+ \; p'^2_y \cdot \left(\sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta)\right)$$
$$+ \; p'_x p'_y \cdot \left(-\cos^2(\theta) + \sin^2(\theta) + \cos^2(\theta) - \sin^2(\theta)\right)$$
$$+ \; p'_x \cdot \left(q'_x\sin(\theta) - q'_y\cos(\theta)\right)$$
$$+ \; p'_y \cdot \left(q'_x\cos(\theta) + q'_y\sin(\theta)\right) = 0$$

Absolute orientation problem in SE(2)

$$\theta^\star = \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left(p'_x\cos(\theta) - p'_y\sin(\theta) - q'_x\right)^2 + \left(p'_x\sin(\theta) + p'_y\cos(\theta) - q'_y\right)^2$$

Derivative: $$\sum_{\mathbf{p}',\mathbf{q}'} 2(p'_x\cos(\theta) - p'_y\sin(\theta) - q'_x)\cdot(-p'_x\sin(\theta) - p'_y\cos(\theta))$$

$$+ \quad 2(p'_x\sin(\theta) + p'_y\cos(\theta) - q'_y)\cdot(p'_x\cos(\theta) - p'_y\sin(\theta)) = 0$$

Simplify: $$\sum_{\mathbf{p}',\mathbf{q}'} \cancel{p'^2_x \cdot \left(-\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta)\right)}$$

$$+ \cancel{p'^2_y \cdot \left(\sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta)\right)}$$

$$+ \cancel{p'_x p'_y \cdot \left(-\cos^2(\theta) + \sin^2(\theta) + \cos^2(\theta) - \sin^2(\theta)\right)}$$

$$+ p'_x \cdot \left(q'_x\sin(\theta) - q'_y\cos(\theta)\right)$$

$$+ p'_y \cdot \left(q'_x\cos(\theta) + q'_y\sin(\theta)\right) = 0$$

Absolute orientation problem in SE(2)

$$\theta^\star = \arg\min_\theta \sum_{\mathbf{p}',\mathbf{q}'} \left(p'_x\cos(\theta) - p'_y\sin(\theta) - q'_x\right)^2 + \left(p'_x\sin(\theta) + p'_y\cos(\theta) - q'_y\right)^2$$

Derivative: $$\sum_{\mathbf{p}',\mathbf{q}'} p'_x \cdot \left(q'_x\sin(\theta) - q'_y\cos(\theta)\right) + p'_y \cdot \left(q'_x\cos(\theta) + q'_y\sin(\theta)\right) = 0$$

Solve: $$\sum_{\mathbf{p}',\mathbf{q}'} p'_x \cdot \left(q'_x\tan(\theta) - q'_y\right) + p'_y \cdot \left(q'_x + q'_y\tan(\theta)\right) = 0$$

$$\sum_{\mathbf{p}',\mathbf{q}'} \tan(\theta) \cdot (p'_xq'_x + p'_yq'_y) + (p'_yq'_x - p'_xq'_y) = 0$$

$$\theta^\star = \arctan\left(\frac{\sum_{\mathbf{p}',\mathbf{q}'} p'_xq'_y - p'_yq'_x}{\sum_{\mathbf{p}',\mathbf{q}'} p'_xq'_x + p'_yq'_y}\right) = \arctan\left(\frac{H_{xy} - H_{yx}}{H_{xx} + H_{yy}}\right)$$

$\mathbf{H} = \sum_i \mathbf{p}'_i\mathbf{q}'^{\top}_i$ … covariance matrix

Absolute orientation problem in **SE(2)**

$$\mathbf{z}^v = \arg\min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p} + \mathbf{t} - \mathbf{q} \right\|^2 = \arg\min_{\mathbf{t},\theta} \sum_i \|\mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i\|^2 + \|\mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2$$

Depends only on θ

Can be always zero by appropriate choice of $\mathbf{t}$

Substitution: $\mathbf{p}'_i = \mathbf{p}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}, \quad \mathbf{q}'_i = \mathbf{q}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

Solution: $\mathbf{H} = \sum_i \mathbf{p}'_i \mathbf{q}'^{\top}_i$ … covariance matrix

$$\theta^\star = \arg\min_\theta \sum_i \|\mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i\|^2 = \arctan\left(\frac{H_{xy} - H_{yx}}{H_{xx} + H_{yy}}\right)$$

$$\mathbf{t}^\star = \arg\min_{\mathbf{t}} \|\mathbf{R}_{\theta^\star}\tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2 = \tilde{\mathbf{q}} - \mathbf{R}_{\theta^\star}\tilde{\mathbf{p}}$$

Absolute orientation problem in **SE(3)**

$$\mathbf{z}^v = \arg\min_{\mathbf{t},\mathbf{R}} \sum_i \| \mathbf{Rp} + \mathbf{t} - \mathbf{q} \|^2 = \arg\min_{\mathbf{t},\mathbf{R}} \sum_i \|\mathbf{Rp}'_i - \mathbf{q}'_i\|^2 + \|\mathbf{R}\tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2$$

Depends only on $\mathbf{R}$

Can be always zero by appropriate choice of $\mathbf{t}$

Substitution: $\mathbf{p}'_i = \mathbf{p}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}, \quad \mathbf{q}'_i = \mathbf{q}_i - \underbrace{\frac{1}{N}\sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

Solution: $\mathbf{H} = \sum_i \mathbf{p}'_i \mathbf{q}'^{\top}_i$ … covariance matrix with SVD decomposition $\mathbf{H} = \mathbf{USV}^\top$

$$\mathbf{R}^\star = \arg\min_{\mathbf{R}} \sum_i \|\mathbf{Rp}'_i - \mathbf{q}'_i\|^2 = \mathbf{VU}^\top$$

$$\mathbf{t}^\star = \arg\min_{\mathbf{t}} \|\mathbf{R}^\star\tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}\|^2 = \tilde{\mathbf{q}} - \mathbf{R}^\star\tilde{\mathbf{p}}$$

Python:

```
H = P @ Q.T
U, S, V = np.linalg.svd(H, full_matrices=True)
```

Summary

- **Static environment + known correspondences** is required assumption
- Given 3D-3D (or 2D-2D) correspondences, **globally optimal alignment in L2 has closed-form solution** (i.e. least-squares solution constrained on SE(3) manifold)
- **Applications:**
  - Lidar-Lidar or Lidar-Robot Calibration
  - Localization from (un)known correspondences
  - Computer graphics for alignment of 3D models
- **Next:** Localization from unknown correspondences ICP

Proof [Arun-TPAMI-87]

$$\mathbf{R}^*, \mathbf{t}^* = \underset{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 =$$

Proof [Arun-TPAMI-87]

$$\mathbf{R}^*, \mathbf{t}^* = \underset{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 =$$

$$= \underset{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}(\mathbf{p}'_i + \widetilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}'_i - \widetilde{\mathbf{q}}\|_2^2 =$$

Proof [Arun-TPAMI-87]

$$\mathbf{R}^*, \mathbf{t}^* = \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 =$$

$$= \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}(\mathbf{p}_i' + \widetilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}_i' - \widetilde{\mathbf{q}}\|_2^2 =$$

$$= \arg\min_{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i' + \underbrace{\mathbf{R}\widetilde{\mathbf{p}} + \mathbf{t} - \widetilde{\mathbf{q}}}_{\mathbf{t}'}\|_2^2 =$$

Proof [Arun-TPAMI-87]

$$\begin{aligned}
\mathbf{R}^*, \mathbf{t}^* &= \underset{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 = \\
&= \underset{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}(\mathbf{p}'_i + \widetilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}'_i - \widetilde{\mathbf{q}}\|_2^2 = \\
&= \underset{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \underbrace{\mathbf{R}\widetilde{\mathbf{p}} + \mathbf{t} - \widetilde{\mathbf{q}}}_{\mathbf{t}'}\|_2^2 = \\
&= \underset{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3}{\arg\min} \sum_i (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}')^\top (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}') =
\end{aligned}$$

Proof [Arun-TPAMI-87]

$$\mathbf{R}^*, \mathbf{t}^* = \underset{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 =$$

$$= \underset{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}(\mathbf{p}_i' + \widetilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}_i' - \widetilde{\mathbf{q}}\|_2^2 =$$

$$= \underset{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i' + \underbrace{\mathbf{R}\widetilde{\mathbf{p}} + \mathbf{t} - \widetilde{\mathbf{q}}}_{\mathbf{t}'}\|_2^2 =$$

$$= \underset{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3}{\arg\min} \sum_i (\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i' + \mathbf{t}')^\top (\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i' + \mathbf{t}') =$$

$$= \underset{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i'\|_2^2 + \underbrace{\sum_i 2(\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i')\mathbf{t}'}_{=0} + \|\mathbf{t}'\|_2^2 =$$

Proof [Arun-TPAMI-87]

$$\mathbf{R}^*, \mathbf{t}^* = \underset{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 =$$

$$= \underset{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}(\mathbf{p}'_i + \widetilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}'_i - \widetilde{\mathbf{q}}\|_2^2 =$$

$$= \underset{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \underbrace{\mathbf{R}\widetilde{\mathbf{p}} + \mathbf{t} - \widetilde{\mathbf{q}}}_{\mathbf{t}'}\|_2^2 =$$

$$= \underset{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3}{\arg\min} \sum_i (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}')^\top(\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}') =$$

$$= \underset{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|_2^2 + \underbrace{\sum_i 2(\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i)\mathbf{t}'}_{=0} + \|\mathbf{t}'\|_2^2 =$$

$$= \underset{\mathbf{R}\in SO(3), \mathbf{t}\in\mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|_2^2 + \|\mathbf{t}'\|_2^2$$

we can reach second term zero by $\mathbf{t} = \widetilde{\mathbf{q}} - \mathbf{R}\widetilde{\mathbf{p}} = \mathbf{t}^*$

Proof [Arun-TPAMI-87]

$$= \underset{\mathbf{R}\in SO(3),\mathbf{t}\in\mathcal{R}^3}{\arg\min} \sum_i \|\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i'\|_2^2 + \|\mathbf{t}'\|_2^2$$

we can reach second term zero by $\mathbf{t} = \tilde{\mathbf{q}} - \mathbf{R}\tilde{\mathbf{p}} = \mathbf{t}^*$

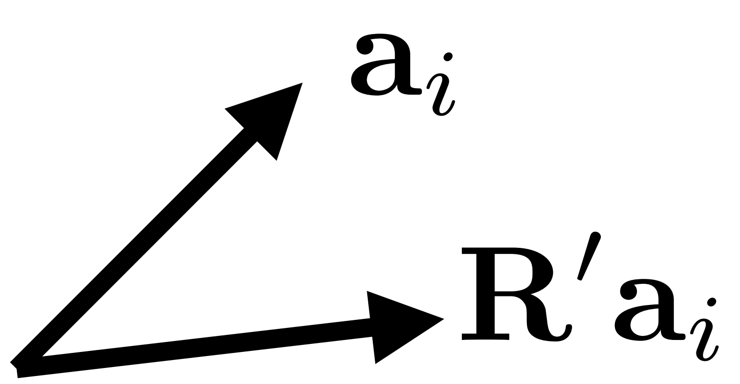
$$\underset{\mathbf{R}\in SO(3)}{\arg\min} \sum_i \|\mathbf{R}\mathbf{p}_i' - \mathbf{q}_i'\|_2^2 = \underset{\mathbf{R}\in SO(3)}{\arg\max} \sum_i \mathbf{q}_i'^\top \mathbf{R}\mathbf{p}_i' =$$

$$= \underset{\mathbf{R}\in SO(3)}{\arg\max} \sum_i \underbrace{\mathbf{q}_i'^\top}_{\mathbf{a}_i} \underbrace{\mathbf{R}\mathbf{p}_i'}_{\mathbf{b}_i} = \underset{\mathbf{R}\in SO(3)}{\arg\max}\ \mathrm{trace}\ \mathbf{R}\underbrace{\mathbf{P}\mathbf{Q}^\top}_{\mathbf{H}} = \mathbf{V}\mathbf{U}^\top$$

$$\underset{\mathbf{R}',\mathbf{R}^*\in SO(3)}{\arg\max}\ \mathrm{trace}\ \mathbf{R}'\mathbf{R}^*\mathbf{U}\mathbf{S}\mathbf{V}^\top$$ … expand into two rotations

$$\underset{\mathbf{R}'\in SO(3)}{\arg\max}\ \mathrm{trace}\ \mathbf{R}' \underbrace{\mathbf{V}\mathbf{U}^\top}_{\mathbf{R}^*} \underbrace{\mathbf{U}\mathbf{S}\mathbf{V}^\top}_{\mathbf{H}} = \underset{\mathbf{R}'\in SO(3)}{\arg\max}\ \mathrm{trace}\ \mathbf{R}' \underbrace{(\mathbf{V}\sqrt{\mathbf{S}})}_{\mathbf{A}} \underbrace{(\sqrt{\mathbf{S}}\mathbf{V})^\top}_{\mathbf{A}^\top} =$$

$$= \underset{\mathbf{R}'\in SO(3)}{\arg\max} \sum_i \mathbf{a}_i^\top \mathbf{R}'\mathbf{a}_i = \mathbf{E}$$



$$\mathrm{trace}\ \mathbf{B}\mathbf{A}^\top = \sum_i \mathbf{a}_i^\top \mathbf{b}_i$$