Quantum Computing

Exercises 9: Quantum Random Walks

1. *At each time step, a quantum walk corresponds to a unitary map $U \in U(N)$ such that*

$$U : \mathcal{H}_G \to \mathcal{H}_G$$
$$|x\rangle \mapsto a|x-1\rangle + b|x\rangle + c|x+1\rangle$$

Show that U is unitary if and only if one of the following three conditions is true: (a) $|a| = 1, b = c = 0$, (b) $|b| = 1, a = c = 0$, (c) $|c| = 1, a = b = 0$.

2. *Demonstrate that the shift operator S, as defined as*

$$S = (|0\rangle\langle 0| \otimes \sum_{x=-\infty}^{+\infty} |x+1\rangle\langle x|) + (|1\rangle\langle 1| \otimes \sum_{x=-\infty}^{+\infty} |x-1\rangle\langle x|)$$

is equivalent to

$$S|i,x\rangle = \begin{cases} |0, x+1\rangle & \text{if } i = 0, \\ |1, x-1\rangle & \text{if } i = 1. \end{cases}$$

3. *Consider a one-dimensional quantum walk on $\mathbb{Z}$ where the coin operator C is parameterized by an angle θ as :*

$$c(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Tue walker starts at $x = 0$ with initial state $|\psi_0\rangle = |i\rangle \otimes |x = 0\rangle$ and $|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Use the shift operator as defined before a) Calculate the state of the system $|\psi_1\rangle$ after 1 step.
b) Compute the probabilities $p(x = 1)$ and $p(x = -1)$ that the walker is at $x = 1$ or $x = -1$.
c) Determine for which values of θ (if they exist) the quantum walks are unbiased ($p(x = 1) = p(x = -1)$)
d) Prove whether the Hadamard walker is biased a not.

4. *In the lecture notes, starting at the state $|\psi_0\rangle = |0\rangle|0\rangle$, we have seen how to obtain the successive states up to $|\psi_3\rangle$ by using the unitary operator $U = S(H \otimes I)$. Derive $|\psi_4\rangle$ for the walker on the finite subset of $\mathbb{Z}$.*

5. *Consider*

$$H_G^{(2)} = \sum_{\omega=1}^{2} \sum_{(i,j)\in E(G)} (|i\rangle \langle j|_\omega + |j\rangle \langle i|_w)$$

where $E(G) = \{(1,2)$ and $(2,1)\}$. This is the Hamiltonian for 2 particles on this G.
a) Assume we have distinguishable walkers. Compute the evolution of the initial state $|\psi_0\rangle = |1,2\rangle$ under the Hamiltonian $H_G^{(2)}$.
b) Assuming the walkers are distinguishable, now compute the evolution for the fermionic state $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|1,2\rangle - |2,1\rangle)$