Sequential decisions under uncertainty
Policy iteration

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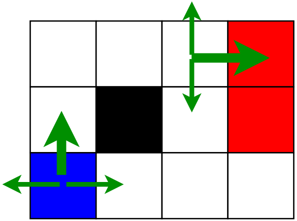
March 28, 2024

Recap: Unreliable actions in observable grid world

- Walls block movement – agent/robot stays in place.
- Actions do not always go as planned.
- Agent receives rewards each time step:
  - Small "living" reward/penalty.
  - Big rewards/penalties at the end.
- Goal: maximize sum of (discounted) rewards



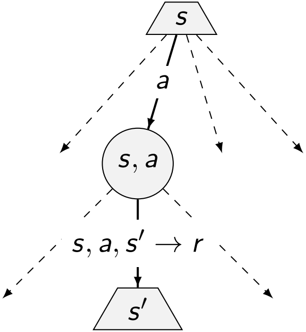
MDPs recap

Markov decision processes (MDPs):

- Set of states $\mathcal{S}$
- Set of actions $\mathcal{A}$
- Transitions $p(s'|s, a)$ or $T(s, a, s')$
- Rewards $r(s, a, s')$; and discount γ

MDP quantities:

- Policy $\pi(s) : \mathcal{S} \to \mathcal{A}$
- Utility – sum of (discounted) rewards.
- Values – expected future utility from a state (max-node), $v(s)$
- Q-Values – expected future utility from a q-state (chance-node), $q(s, a)$



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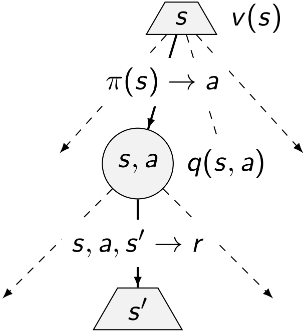


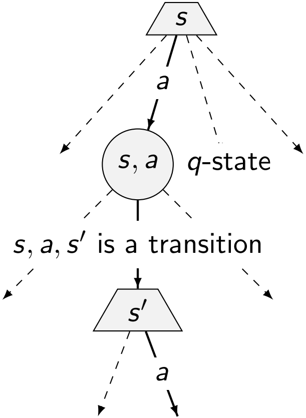
Optimal quantities

- The optimal policy: $\pi^*(s)$ – optimal action from state s
- Expected utility/return of a policy.

$$U^\pi(S_t) = \mathrm{E}^\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1}\right]$$

Best policy π^* maximizes above.

- The value of a state s: $v^*(s)$ – expected utility starting in s and acting optimally.
- The value of a q-state (s, a): $q^*(s, a)$ - expected utility having taken a from state s and acting optimally thereafter.



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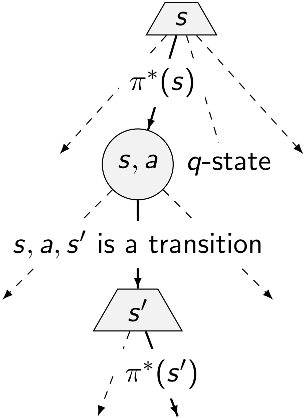

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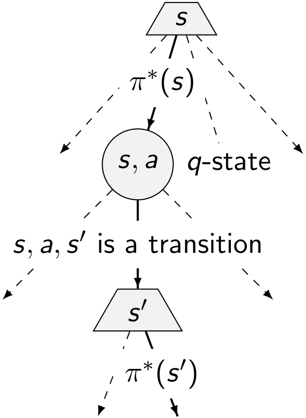


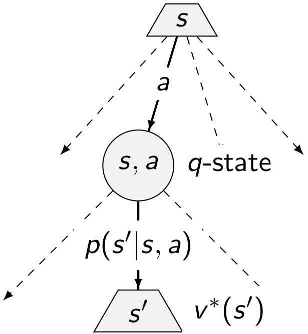
v^* and q^*

The value of a q-state (s, a):

$$q^*(s,a) = \sum_{s'} p(s'|a,s)\left[r(s,a,s') + \gamma\, v^*(s')\right]$$

The value of a state s:

$$v^*(s) = \max_a q^*(s,a)$$



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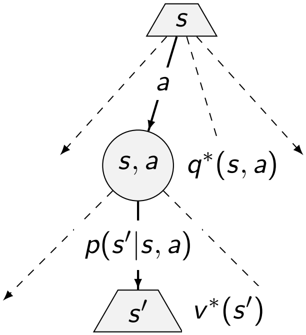

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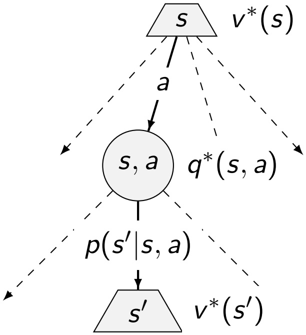
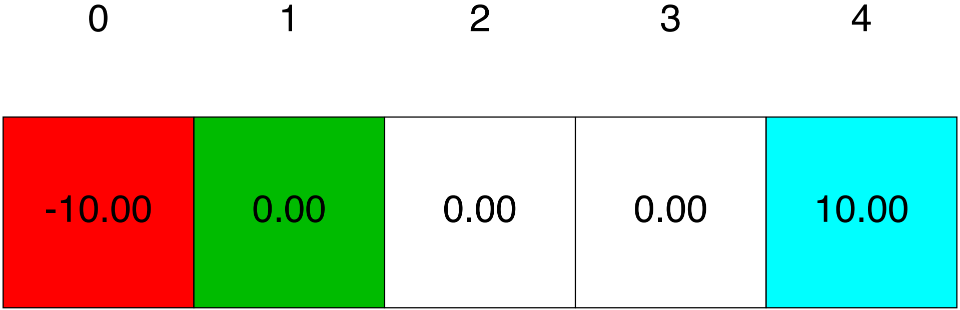


Maze: $V_0 = [0, 0, 0]^\top$, $r(s) = -1$, deterministic robot, $\mathcal{A} = \{\leftarrow, \uparrow, \downarrow, \rightarrow\}$, $\gamma = 1$

0	1	2	3	4
-10.00	0.00	0.00	0.00	10.00

$$
\begin{aligned}
q^*(s, a) &= \sum_{s'} p(s'|a, s)\left[r(s, a, s') + \gamma\, v^*(s')\right] \\
v^*(s) &= \max_a q^*(s, a)
\end{aligned}
$$

What will be V^* after first sweep? $V_1^* = [v_1^*(1), v_1^*(2), v_1^*(3)]^\top$?



Sweep is meant as the Bellmann update for all states: $V_1^* = BV_0^*$. $r(s) = -1$. Assume sync version of the algorithm.

A: $V_1^* = [-1, -1, 9]^\top$

B: $V_1^* = [0, 8, 9]^\top$

C: $V_1^* = [-1, 0, 0]^\top$

D: $V_1^* = [-11, 8, 9]^\top$

What will be V^* after second sweep? $V_2^* = [v_2^*(1), v_2^*(2), v_2^*(3)]^\top$?

0	1	2	3	4
-10.00	0.00	0.00	0.00	10.00

Sweep is meant as the Bellmann update for all states: $V_2^* = B(BV_0^*)$. $r(s) = -1$. Assume sync version of the algorithm.

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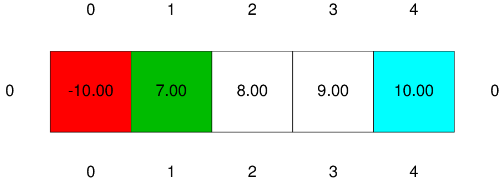
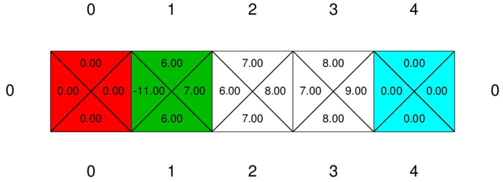
B: $V_2^* = [-1, 8, 9]^\top$

C: $V_2^* = [-2, 8, 9]^\top$

D: $V_2^* = [7, 8, 9]^\top$

Maze: v^* vs. q^*, deterministic robot, $\mathcal{A} = \{\leftarrow, \uparrow, \downarrow, \rightarrow\}$





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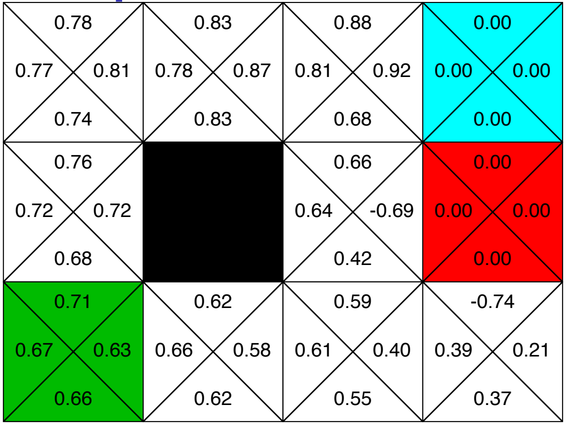
\end{aligned}
$$

Maze: v^* vs. q^*, $\gamma = 1$, $T = [0.8, 0.1, 0.1, 0]$

0.81	0.87	0.92	1.00
0.76		0.66	-1.00
0.71	0.66	0.61	0.39



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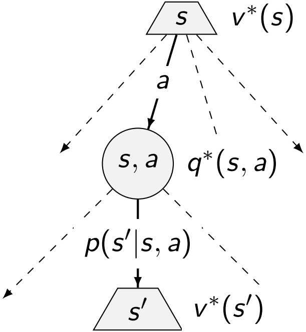
Value iteration

- Bellman equations **characterize** the optimal values

$$v^*(s) = \max_{a \in A(s)} \sum_{s'} p(s'|s,a) \left[r(s,a,s') + \gamma v^*(s') \right]$$

- Value iteration **computes** them:

$$V_{k+1}(s) \leftarrow \max_{a \in A(s)} \sum_{s'} p(s'|s,a) \left[r(s,a,s') + \gamma V_k(s') \right]$$

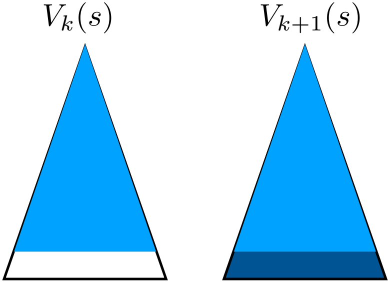


Value iteration is a fixed point solution method.

Convergence

$$V_{k+1}(s) \leftarrow \max_{a \in A(s)} \sum_{s'} p(s'|s,a) \left[r(s,a,s') + \gamma V_k(s') \right]$$

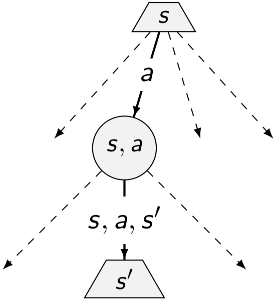
- Thinking about special cases: deterministic world, $\gamma = 0$, $\gamma = 1$.
- For all s, $V_k(s)$ and $V_{k+1}(s)$ can be seen as expectimax search trees of depth k and $k+1$



- Bottom (last) layer, zeros for $V_k(s)$, true rewards for $V_{k+1}(s)$
- Last layer $\langle R_{min}, R_{max} \rangle$
- But the last layer is γ^k discounted ...
- hence, V_k and V_{k+1} are no more than $\gamma^k \max |R|$ apart.
- The k increases, the values converge.

From Values to Policy

Policy extraction - computing actions from Values



- Assume we have $v^*(s)$
- What is the optimal action?
- We need a one-step expectimax:

	0	1	2	3	
0	0.81	0.87	0.92	1.00	0
1	0.76		0.66	-1.00	1
2	0.71	0.66	0.61	0.39	2
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$$\pi^*(s) = \arg\max_{a \in \mathcal{A}(s)} \sum_{s'} p(s' \mid s, a) \left[r(s, a, s') + \gamma v^*(s') \right]$$

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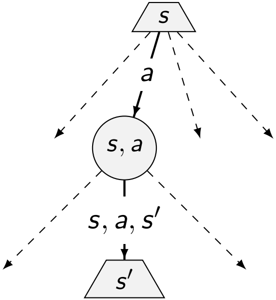

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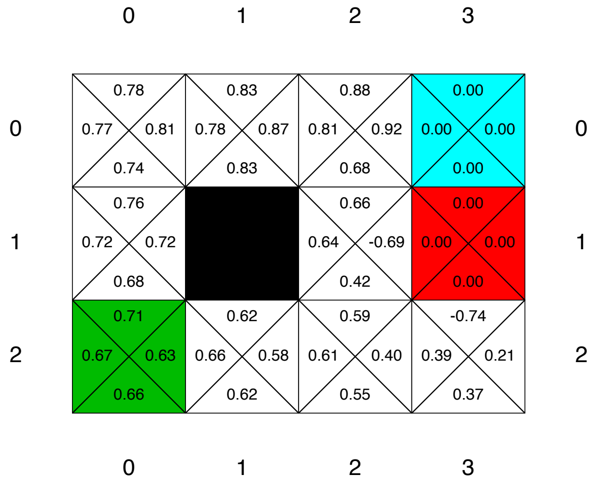
Policy extraction - computing actions from q-Values

- Assume we have $q^*(s, a)$
- What is the optimal action?
- Just take the (arg) max:

$$\pi^*(s) = \arg\max_{a \in \mathcal{A}(s)} q^*(s, a)$$

Actions are easier to extract from q-values.



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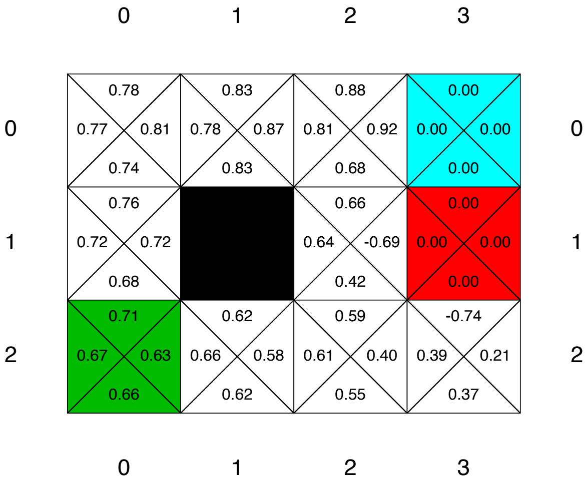


What is wrong with the Value iteration?

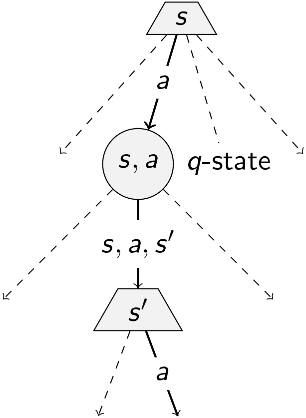
$$V_{k+1}(s) \leftarrow \max_{a \in \mathcal{A}(s)} \sum_{s'} p(s' \mid s, a) \left[r(s, a, s') + \gamma V_k(s') \right]$$

- What is complexity of one iteration - over all S states?
- When does the iteration stop?
- When the does the policy converge?
- Can we compute the policy directly?

Policy evaluation

- Assume $\pi(s)$ given.
- How to evaluate (compare)?

Fixed policy, do what π says



- Expectimax trees "max" over all actions . . .
- Fixed π for each state $\rightarrow$ no "max" operator!

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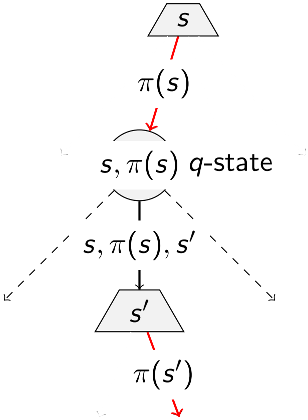

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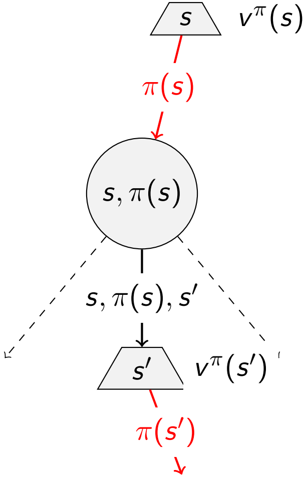
State values under a fixed policy



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$$v^\pi(s) = \sum_{s'} p(s' \mid s, \pi(s))\big[r(s, \pi(s), s') + \gamma v^\pi(s')\big]$$

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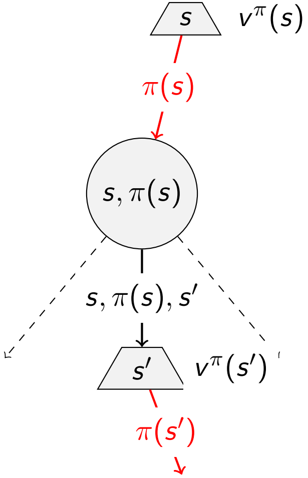

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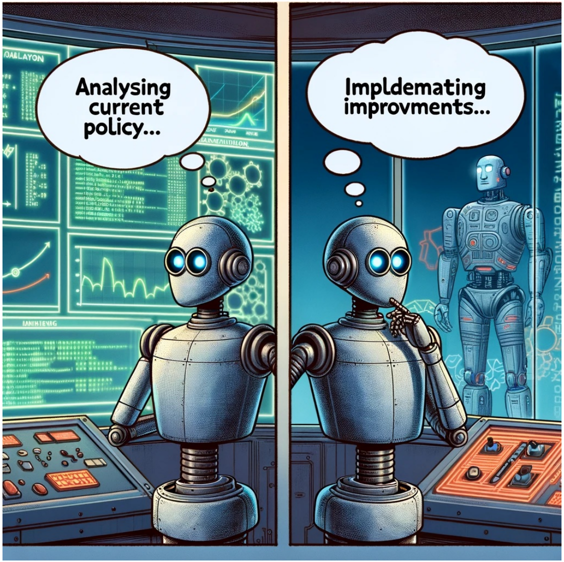
Finding the best policy directly by the *Policy iteration* method

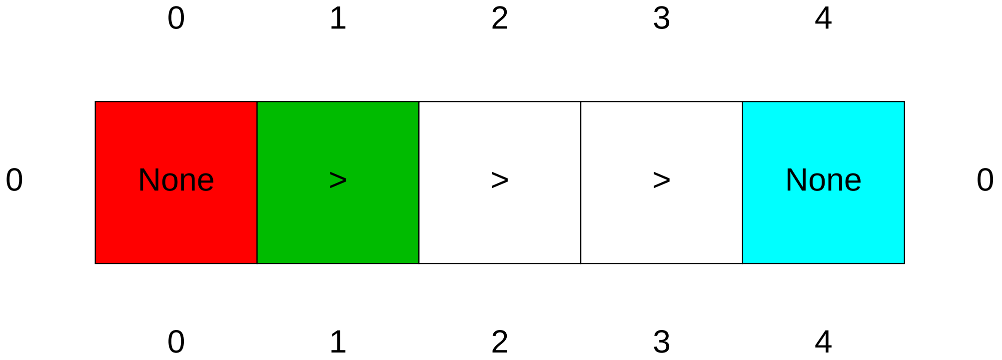
- Start with a random policy.
- Step 1: Evaluate it.
- Step 2: Improve it.
- Repeat steps until policy converges.



How to evaluate policy? Policy determines state values

$$v^{\pi}(s) = \sum_{s'} p(s' \mid s, \pi(s)) \left[r(s, \pi(s), s') + \gamma v^{\pi}(s') \right]$$

Case: $\gamma = 1$ and deterministic robot. What are $V^{\pi}(1), V^{\pi}(2), V^{\pi}(3)$?



Policy iteration - equations

- Policy π evaluation. Solve equations or iterate until convergence.

$$V^{\pi_i}_{k+1}(s) \leftarrow \sum_{s'} p(s' \mid s, \pi(s)) \left[r(s, \pi(s), s') + \gamma V^{\pi_i}_k(s') \right]$$

- Policy improvement. Look-ahead and keep optimality. Policy extraction from fixed values.

$$\pi_{i+1}(s) = \arg\max_{a \in \mathcal{A}(s)} \sum_{s'} p(s' \mid s, a) \left[r(s, a, s') + \gamma V^{\pi_i}_k(s') \right]$$

Policy iteration - a problem(?)

$$v^\pi(s) = \sum_{s'} p(s' \mid s, \pi(s)) \left[r(s, \pi(s), s') + \gamma v^\pi(s') \right]$$

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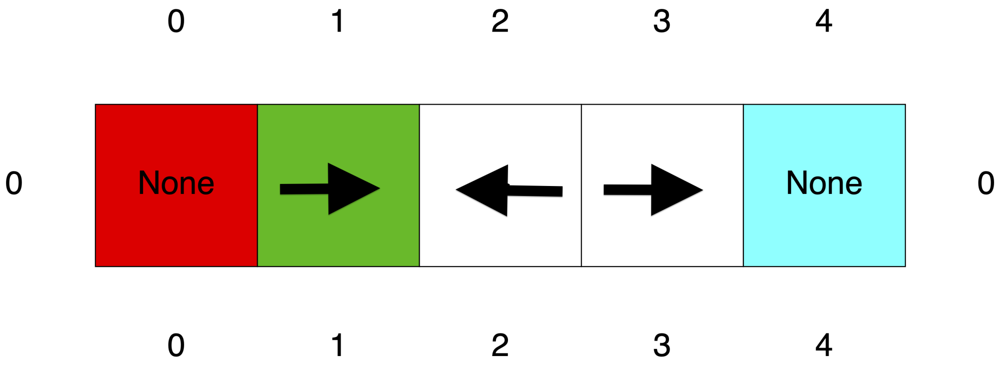

Policy iteration algorithm

function POLICY-ITERATION(env) **returns:** policy π
  input: env - MDP problem
  $\pi(s) \leftarrow$ random $a \in A(s)$ in all states
  $V(s) \leftarrow 0$ in all states
  repeat ▷ iterate values until no change in policy
    $V \leftarrow$ POLICY-EVALUATION(π, V, env)
    unchanged $\leftarrow$ True
    for each state s **in** S **do**
      if $\max\limits_{a \in A(s)} \sum_{s'} P(s'|a,s)V(s') > \sum_{s'} P(s'|s,\pi(s))V(s')$ **then**
        $\pi(s) \leftarrow \arg\max\limits_{a \in A(s)} \sum_{s'} P(s'|a,s)V(s')$
        unchanged $\leftarrow$ False
  until unchanged

Policy vs. Value iteration

- Value iteration.
  - Iteration updates values and policy. (policy only implicitly – can be extracted from values)
  - No track of policy.
- Policy iteration.
  - Update of values is faster – only one action per state.
  - New policy from values (slower).
  - New policy is better or done.
- Both methods belong to Dynamic programming realm.
- Think (compare) complexities of one sweep (iteration step)

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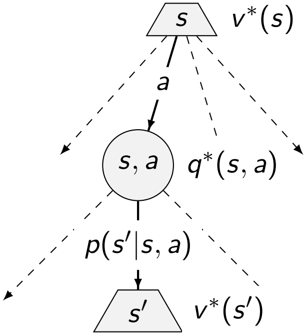
Value/policy iteration (dynamic programming) vs. direct search

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} p(s'|s,a) V_k(s')$$

- value/policy iteration is an *off-line* method
- direct (expectimax) search is an *on-line* method
- sometimes too many states, ...
- but for γ close to 1 the tree is too deep
- we will learn about approximate methods (RL)



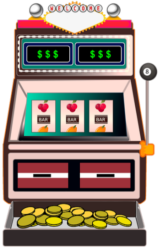
References

Further reading: Chapter 17 of [1] however, policy iteration is quite compact there. More detailed discussion can be found in chapter Dynamic programming in [2] with slightly different notation, though. This lecture has been also greatly inspired by the 9th lecture of CS 188 at http://ai.berkeley.edu as it convincingly motivates policy search and offers an alternative convergence proof of the value iteration method.

[1] Stuart Russell and Peter Norvig.
Artificial Intelligence: A Modern Approach.
Prentice Hall, 3rd edition, 2010.
http://aima.cs.berkeley.edu/.

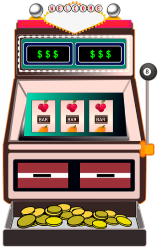
[2] Richard S. Sutton and Andrew G. Barto.
Reinforcement Learning; an Introduction.
MIT Press, 2nd edition, 2018.
http://www.incompleteideas.net/book/the-book-2nd.html.

(Multi-armed) Bandits



$p(s'|s,a)$ and $r(s,a,s')$ not known!

(Multi-armed) Bandits



$p(s'|s, a)$ and $r(s, a, s')$ not known!

10 armed bandit, what arm to pull?