Elements of Geometry for Computer Vision and Computer Graphics

Translation of Euclid's Elements by Adelardus Bathensis (1080–1152)

2021 Lecture 2

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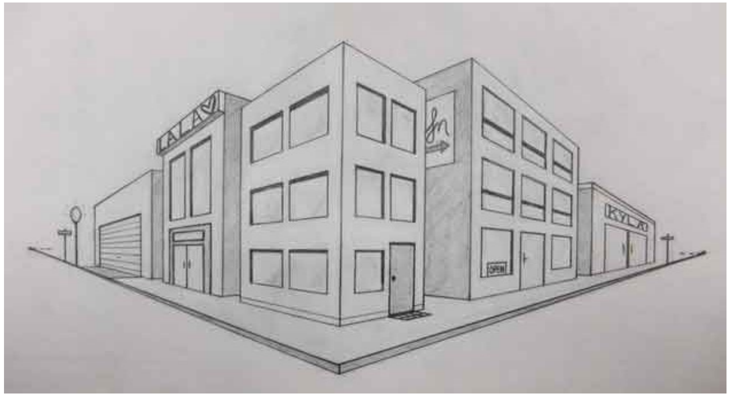
5 Perspective camera

Lines in space project to lines in images



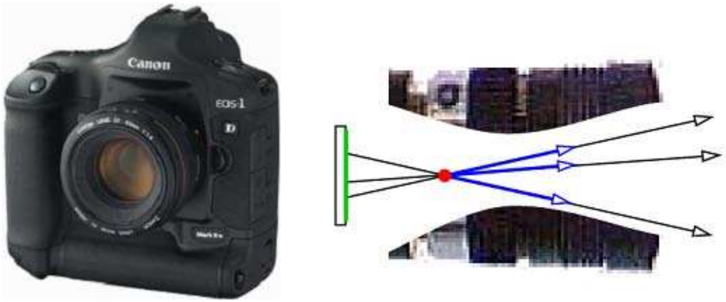
Modern photographic camera, Figure 5.1, is an interesting and advanced device. We shall abstract from all physical and technical details of image formation and will concentrate solely on its geometry. From the point of view of geometry, a perspective camera projects point X from space into an image point x by intersecting the line connecting X with the *projection* center (red) and a planar image plane (green), Figure 5.1(b).

5.1 Perspective camera model

Let us now develop a mathematical model of the perspective camera. The model will allow us to project space point X into image point x and to find the ray p in space along the which point X has been projected.

§ 1 Camera coordinate system Figure 5.2 shows the geometry of the perspective camera. Point X is projected along ray p from three-dimensional space to point x into two-dimensional image. Point x is obtained as the intersection of ray p with planar *image plane* π. Ray p is constructed by joining point X with the *projection center* C. The plane through the projection center C, which is parallel to the image plane is called the *principal plane*.

The image plane is equipped with an image coordinate system (§ 1), (o, α), where o is the origin and $\alpha = [\vec{b}_1, \vec{b}_2]$ is the basis of the image coordinate system. Notice that the basis α is shown as non-orthogonal. We want to develop a general camera model, which will be applicable



(a) (b)

Figure 5.1: Perspective camera (a) is geometrically a point (red) and an image plane (green) (b).

even in the situation when image coordinate system is not rectangular. Point x is represented by vector $\vec{u}$ in (o, α)

$$\vec{u} = u\,\vec{b}_1 + v\,\vec{b}_2 \qquad \text{i.e.} \qquad \vec{u}_\alpha = \begin{bmatrix} u \\ v \end{bmatrix} \tag{5.1}$$

Three-dimensional space is equipped with a *world coordinate system* (O, δ), where O is the origin and $\delta = [\vec{d}_1, \vec{d}_2, \vec{d}_3]$ is a three-dimensional orthonormal basis. Point X is represented by vector $\vec{X}$ in (O, δ). The camera projection center is represented by vector $\vec{C}$ in (O, δ).

Let us next define the *camera coordinate system*. The system will be derived from the image coordinate system to make the construction of coordinates of the direction vector $\vec{x}$ of p extremely simple.

Camera coordinate system (C, β) has the origin in the projection center C and its basis $\beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3]$ is constructed by re-using the two basis vectors of α and adding the third basic vector $\vec{b}_3$, which corresponds to vector $\overrightarrow{Co}$. We see that vectors in β form a basis when point C is not in π, which is

Perspective camera

- Lines from space are projected to lines in images

Pinhole optics

- All rays go throug a single pinhole

Camera obscura

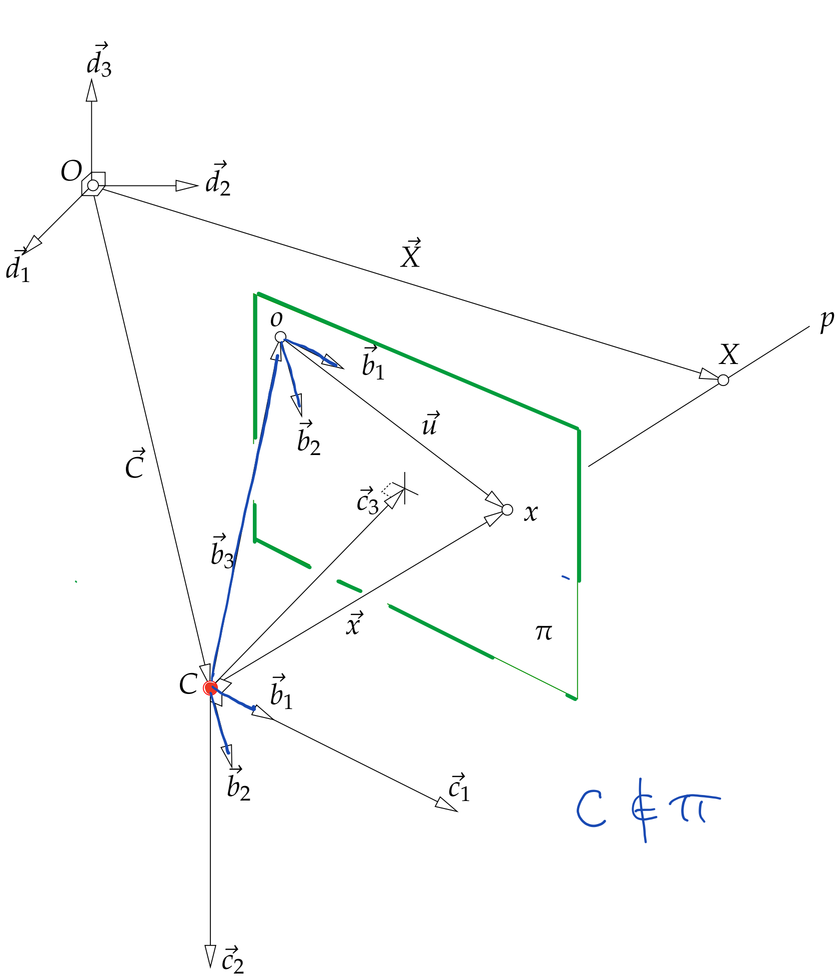


Figure 5.2: Coordinate systems of perspective camera.

World Coordinate system

(O, δ) $\quad \delta = [\vec{d}_1, \vec{d}_2, \vec{d}_3]$ $\quad \vec{X}_\delta$

Image Coordinate System

(o, α) $\quad \alpha = [\vec{b}_1, \vec{b}_2]$

$$\vec{u} = u\,\vec{b}_1 + v\,\vec{b}_2 \quad \text{i.e.} \quad \vec{u}_\alpha = \begin{bmatrix} u \\ v \end{bmatrix}$$

Camera Coordinate System

(C, β) $\quad \beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3] \leftarrow$ derived from α

$$\vec{x} = \vec{u} + \vec{b}_3 = u\,\vec{b}_1 + v\,\vec{b}_2 + 1\,\vec{b}_3$$

$$\vec{x}_\beta = \vec{x}_{[\vec{b}_1, \vec{b}_2, \vec{b}_3]} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix}$$

satisfied for every meaningful perspective camera. Notice also that the camera coordinate system is three-dimensional.

Image points o and x are in plane π, which is in three-dimensional space, and therefore we can consider them as points of that space too. Point x is in (C, β) represented by vector $\vec{x}$, which is the direction vector of the projection ray p along which point X has been projected into x. We see that vectors $\vec{u}$, $\vec{x}$, $\vec{b}_3$ form a triangle such that

$$\begin{aligned} \vec{x} &= \vec{u} + \vec{b}_3 && (5.2) \\ &= u\,\vec{b}_1 + v\,\vec{b}_2 + 1\,\vec{b}_3 && (5.3) \end{aligned}$$

and therefore

$$\vec{x}_\beta = \vec{x}_{[\vec{b}_1, \vec{b}_2, \vec{b}_3]} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix}. \tag{5.4}$$

Notice that basis β has been constructed in a very special way to facilitate construction of $\vec{x}_\beta$. We can use u, v directly since β re-uses vectors of α and the third coordinate is always 1 by the construction of $\vec{b}_3$. Although we do not know exact position of C w.r.t. the image plane, we know that it is not in the plane π and hence a meaningful camera coordinate system constructed this way exists.

Notice next that the camera coordinate system is right-handed. This is because when looking at a scene from a point C through the image plane, the image is constructed by intersecting image rays with the image plane, which is in front and hence the vector $\vec{b}_3$ points towards the scene. We see that vectors of β form a right-handed system.

Let us mention that we have used deeper properties of linear and affine spaces. In particular, we were making use of the concept of *free vector* in the following way. We look at vectors $\vec{b}_1$, $\vec{b}_2$ and $\vec{u}$ as on a free vectors. Therefore, coordinates of the representative of $\vec{u}$ beginning in o with respect to representatives of $\vec{b}_1$, $\vec{b}_2$ beginning in o equal the coordinates of the representative of $\vec{u}$ beginning in C with respect to representatives of $\vec{b}_1, \vec{b}_2$ beginning in C. Hence u, v reappear as the first two coordinates of $\vec{x}$.

For usual consumer cameras, vector $\vec{b}_3$ is often much longer than vectors $\vec{b}_1, \vec{b}_2$ and often not orthogonal to them. Therefore, basis β is in general neither orthonormal nor orthogonal! This has severe consequences since we can't measure angles and distances in the space using β, unless we find out what are the lengths of its vectors and what are the angles between them.

§ **2 Perspective projection** Point X has been projected along p into x. Since $\vec{x}$ is a direction vector of p, point X can be represented in (C, β) by

$$\eta \, \vec{x} \tag{5.5}$$

for some real *non-negative*[1] η. The value of η corresponds to the scaled depth of X, i.e. the distance of X from the plane passing through C and generated by $\vec{b}_1$, $\vec{b}_2$ in units equal to the distance of C from π. Value η is not known since it "has been lost" in the process of projection[2] but will serve us to parametrize the projection ray in order to get coordinates of all possible points in space that could project into x.

Let us now relate the coordinates $\vec{u}_\alpha$, which are measured in the image, to the coordinates $\vec{X}_\delta$, which are measured in the world coordinate system. First consider vectors $\vec{X}$, $\vec{C}$ and $\vec{x}$. They are coplanar and we see that there holds

$$\eta \, \vec{x} \;=\; \vec{X} - \vec{C} \tag{5.6}$$

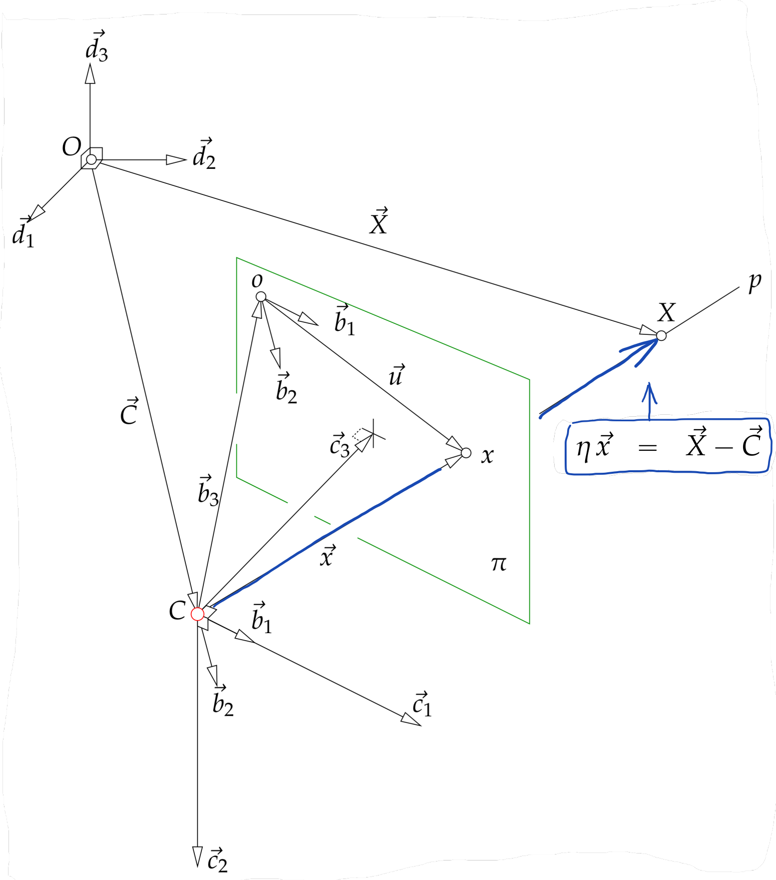
[1] Here we choose $\vec{x}$ such that η is non-negative. Considering negative η, as in [13], may be necessary if it is not clear how has the image coordinate systems been defined or how has $\vec{x}$ been chosen. For instance, if $\vec{x}$ has been chosen to point along ray p away from X, η would have to be negative.

[2] It can be recovered when a point X is observed by two cameras with different projection centers.

To pass to coordinates, we will use the camera coordinate system, in which we can write

vectors
$$\boxed{\eta\,\vec{x} = \vec{X} - \vec{C}}$$

in coordinates β
$$\eta\,\vec{x}_\beta = \vec{X}_\beta - \vec{C}_\beta \tag{5.7}$$
$$\eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix} = \vec{X}_\beta - \vec{C}_\beta \tag{5.8}$$



Next we shall pass to the coordinates w.r.t. basis δ on the right hand side of Equation 5.8 by introducing a matrix $\mathtt{A}$, which transforms coordinates of a general vector $\vec{y}$ from basis δ to basis β, i.e.

$$\vec{y}_\beta = \mathtt{A}\,\vec{y}_\delta \tag{5.9}$$

We know from linear algebra (§ 3) that such a matrix exists. We write

$$\eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix} = \underline{\mathtt{A}\,(\vec{X}_\delta - \vec{C}_\delta)}$$

$\mathtt{A} \in \mathbb{R}^{3\times 3}$

$$\eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix} = \mathtt{A} \underline{\begin{bmatrix} \mathtt{I} \mid -\vec{C}_\delta \end{bmatrix}} \underline{\begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}} \tag{5.10}$$
$$\underline{\eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix}} = \mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{5.11}$$
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \leftarrow \begin{bmatrix} \eta u \\ \eta v \\ \eta \end{bmatrix}$$
$$\eta\,\vec{x}_\beta = \underline{\mathtt{P}_\beta} \underline{\begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}} \approx X \tag{5.12}$$

with 3×4 *image projection matrix*

$$\mathtt{P}_\beta = \begin{bmatrix} \underset{3\times 3}{\mathtt{A}} & \mid \underset{3\times 1}{-\mathtt{A}\,\vec{C}_\delta} \end{bmatrix} \longrightarrow \in \mathbb{R}^{3\times 4} \tag{5.13}$$

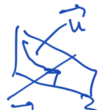
$$u = \frac{a}{c}$$
$$v = \frac{b}{c}$$

§ 3 Projection equation Equation 5.11 describes the relationship between measurement $\vec{u}_\alpha$ in the image and measurement $\vec{X}_\delta$ in space. It says that $\vec{X}_\delta$ is projected into $\vec{u}_\alpha$ since there exists η such that Equation 5.11

holds. Notice that η multiple of the vector on the left of Equation 5.11 is obtained by a linear mapping represented by matrix $\mathtt{P}_\beta$ from vector $\vec{X}_\delta$ on the right.

When computing $\vec{u}_\alpha$ from $\vec{X}_\delta$, we actually eliminate η using the last row of the (matricidal) equation (5.11)

$$\vec{u}_\alpha = \begin{bmatrix} \frac{\mathtt{p}_1^\top \mathtt{X}}{\mathtt{p}_3^\top \mathtt{X}} \\ \frac{\mathtt{p}_2^\top \mathtt{X}}{\mathtt{p}_3^\top \mathtt{X}} \end{bmatrix} \tag{5.14}$$



$\vec{x}$ $\mathbb{R}^3 \to \mathbb{R}^2$

$\vec{x}_\beta \to \vec{u}_\alpha$

where we introduced rows of $\mathtt{p}_1, \mathtt{p}_2, \mathtt{p}_3$ of $\mathtt{P}$ and a 4×1 vector $\mathtt{X}$ as follows

$$\mathtt{P}_\beta = \begin{bmatrix} \mathtt{p}_1^\top \\ \mathtt{p}_2^\top \\ \mathtt{p}_3^\top \end{bmatrix} \quad \text{and} \quad \mathtt{X} = \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{5.15}$$

Notice that the projection equation is not linear. It is a rational function of the first order polynomials in elements of $\mathtt{X}$.

§ 4 Projection ray Having an image point $\vec{u}_\alpha$, we can construct its projection ray p in space. The ray consists of all points $\vec{Y}$ that can project to $\vec{u}_\alpha$. In (C, β), the ray is emanating from the origin C. We parametrize it by real η and express it in (O, δ) by vector $\vec{X}_\delta$

$$\begin{aligned} \vec{Y}_\beta &= \eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix} = \eta\, \vec{x}_\beta \\ \vec{X}_\delta &= \eta\, \mathtt{A}^{-1} \vec{x}_\beta + \vec{C}_\delta \end{aligned} \tag{5.16}$$

Notice that $\vec{X}_\delta$ (5.16) can also be obtained for a given η by solving the system of linear equations (5.12) for $\vec{X}_\delta$.

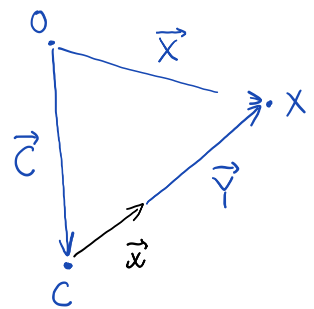
<u>Image projection is</u>

<u>a rational function</u>

$$X = \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P_\beta = \begin{bmatrix} p_1^T \\ p_2^T \\ p_3^T \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix}$$

$$\vec{u}_\alpha = \begin{bmatrix} \frac{p_1^T X}{p_3^T X} \\ \frac{p_2^T X}{p_3^T X} \end{bmatrix} = \begin{bmatrix} \frac{p_{11}x + p_{12}y + p_{13}z + p_{14}}{p_{31}x + p_{32}y + p_{33}z + p_{34}} \\ \frac{p_{21}x + p_{22}y + p_{23}z + p_{24}}{p_{31}x + p_{32}y + p_{33}z + p_{34}} \end{bmatrix}$$



Quiz

q^{21}

5.2 Computing image projection matrix from images of six points

Let us now consider the task of finding the $\mathtt{P}_\beta$ from measurements. We shall consider the situation when we can measure points in space as well as their projection in the image. Consider a pair of such measurements $[x, y, z]^\top \overset{corr}{\leftrightarrow} [u, v]^\top$. There holds

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathtt{Q} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathtt{Q}\,\mathtt{X} \tag{5.17}$$

for some real λ, 3×4 matrix $\mathtt{Q}$ and 4×1 coordinate vector $\mathtt{X}$. Notice that we introduced new symbols λ and $\mathtt{Q}$ to emphasize that they are determined by Equation 5.17 up to a non-zero scale

$$\boxed{\mathtt{Q} = \xi\, \mathtt{P}_\beta} \tag{5.18}$$

We will see that this will have further consequences.

Introduce symbols for rows of $\mathtt{Q}$

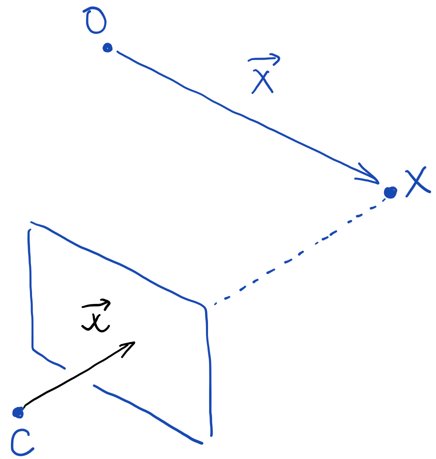
$$\mathtt{Q} = \begin{bmatrix} \mathtt{q}_1^\top \\ \mathtt{q}_2^\top \\ \mathtt{q}_3^\top \end{bmatrix} \tag{5.19}$$

and rewrite the above matrix equation as

$$\lambda\, u = \mathtt{q}_1^\top \mathtt{X} \tag{5.20}$$

$$\lambda\, v = \mathtt{q}_2^\top \mathtt{X} \tag{5.21}$$

$$\lambda = \mathtt{q}_3^\top \mathtt{X} \tag{5.22}$$



$$\lambda \vec{x}_\beta = P_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}}$$

$$\downarrow$$

$$\xi P_\beta = Q$$

Eliminate λ from the first two equations using the third one

$$(\mathtt{q}_3^\top \mathtt{X})\, u = \mathtt{q}_1^\top \mathtt{X} \tag{5.23}$$
$$(\mathtt{q}_3^\top \mathtt{X})\, v = \mathtt{q}_2^\top \mathtt{X} \tag{5.24}$$

move all to the left hand side and reshape it using $\mathtt{x}^\top \mathtt{y} = \mathtt{y}^\top \mathtt{x}$

$$\mathtt{X}^\top \mathtt{q}_1 - (u\, \mathtt{X}^\top)\, \mathtt{q}_3 = 0 \tag{5.25}$$
$$\mathtt{X}^\top \mathtt{q}_2 - (v\, \mathtt{X}^\top)\, \mathtt{q}_3 = 0 \tag{5.26}$$

Introduce vector of parameters (which are elements of $\mathtt{Q}$)

$$\mathtt{q} = [\,\mathtt{q}_1^\top \;\; \mathtt{q}_2^\top \;\; \mathtt{q}_3^\top\,]^\top \in \mathbb{R}^{12\times 1} \tag{5.27}$$

and express the above two equations in matrix form

$$\begin{bmatrix} \overbrace{x & y & z & 1}^{X^\top} & 0 & 0 & 0 & 0 & \overbrace{-u\,x & -u\,y & -u\,z & -u}^{-uX^\top} \\ 0 & 0 & 0 & 0 & \underbrace{x & y & z & 1}_{X^\top} & \underbrace{-v\,x & -v\,y & -v\,z & -v}_{-vX^\top} \end{bmatrix} \mathtt{q} = 0$$
$$\mathtt{M}\;\; \mathtt{q} = 0 \tag{5.28}$$

Every correspondence $[x, y, z]^\top \overset{corr}{\leftrightarrow} [u, v]^\top$ brings two rows into the matrix $\mathtt{M}$ (5.28). We need therefore at least 6 correspondences in general position to obtain 11 linearly independent rows in Equation 5.28 to obtain a one-dimensional space of solutions.

If $\mathtt{Q}$ is a solution to Equation 5.28, then $\tau\, \mathtt{Q}$ is also a solution and both determine the same projection for any positive τ since

$$(\tau\, \mathtt{Q})\; \mathtt{X} = \tau\, (\mathtt{Q}\, \mathtt{X}) = \tau\, (\lambda\, \vec{x}_\beta) = (\tau \lambda)\, \vec{x}_\beta \tag{5.29}$$

Assuming $\mathtt{P}_\beta = \tau\, \mathtt{Q}$ leads to $\lambda = \eta/\tau$. We see that we can't recover $\mathtt{P}_\beta$ but only its non-zero multiple. Therefore, when solving Equation 5.28, we are looking for one-dimensional subspace of 3×4 matrices of *rank* 3. Such a subspace determines one projection. Also note that the zero matrix does not represent any interesting projection.

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \in \mathbb{R}^{12\times 1}$$

$Mq = 0$ — Exact: rank $M = 11$; Noisy: rank $M = 12$

$n \cdot 2$ rows $\times$ 12 cols

$$\underbrace{\begin{bmatrix} x_1 & y_1 & z_1 & 1 & 0 & 0 & 0 & 0 & -u_1 x_1 & -u_1 y_1 & -u_1 z_1 & -u_1 \\ 0 & 0 & 0 & 0 & x_1 & y_1 & z_1 & 1 & -v_1 x_1 & -v_1 y_1 & -v_1 z_1 & -v_1 \\ & & & & & & \vdots & & & & & \\ x_6 & y_6 & z_6 & 1 & 0 & 0 & 0 & 0 & -u_6 x_6 & -u_6 y_6 & -u_6 z_6 & -u_6 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{x} & \cancel{y} & \cancel{z} & \cancel{1} & \cancel{vx} & \cancel{vy} & \cancel{vz} & \cancel{v} \end{bmatrix}}_{M' \in \mathbb{R}^{11\times 12}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}}_{q \in \mathbb{R}^{12\times 1}} = 0$$

rank $M' = 11$

$\neq 0$

$q \longrightarrow Q \approx P_\beta$

1D sols

Notice that when considering more correspondences, $\mathtt{M}$ becomes

$$\mathtt{M}\,\mathtt{q} = \begin{bmatrix} x_1 & y_1 & z_1 & 1 & 0 & 0 & 0 & 0 & -u_1x_1 & -u_1y_1 & -u_1z_1 & -u_1 \\ x_2 & y_2 & z_2 & 1 & 0 & 0 & 0 & 0 & -u_2x_2 & -u_2y_2 & -u_2z_2 & -u_2 \\ & & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & x_1 & y_1 & z_1 & 1 & -v_1x_1 & -v_1y_1 & -v_1z_1 & -v_1 \\ 0 & 0 & 0 & 0 & x_2 & y_2 & z_2 & 1 & -v_2x_2 & -v_2y_2 & -v_2z_2 & -v_2 \\ & & & & & & \vdots & & & & & \end{bmatrix} \mathtt{q} = 0 \tag{5.30}$$

11 x 12

Matrix $\mathtt{M}$ can be more concisely rewritten as

$$\mathtt{M} = \begin{bmatrix} \mathtt{X}_1^\top & \mathtt{0}^\top & -u_1\,\mathtt{X}_1^\top \\ \mathtt{X}_2^\top & \mathtt{0}^\top & -u_2\,\mathtt{X}_2^\top \\ & \vdots & \\ \mathtt{0}^\top & \mathtt{X}_1^\top & -v_1\,\mathtt{X}_1^\top \\ \mathtt{0}^\top & \mathtt{X}_2^\top & -v_2\,\mathtt{X}_2^\top \\ & \vdots & \end{bmatrix} \tag{5.31}$$

with $\mathtt{0}^\top = [0, 0, 0, 0]$.

Quiz

q^{22}

§ 1 A more general procedure for computing $\mathtt{Q}$ We shall next develop and alternative formulation for finding matrix $\mathtt{Q}$. Let us come back to Equation 5.17

$$\lambda\,\vec{u} = \mathtt{Q}\,\mathtt{X} \tag{5.32}$$

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathtt{Q} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathtt{Q}\,\mathtt{X}$$

Above, we have eliminated λ assuming $\vec{u}_3 = 1$. Let us now present an alternative procedure for eliminating λ, which works for any non-zero $\vec{u} = [u, v, w]^\top$, i.e. even when $w = 0$. The trick is to realize that

$$\vec{0} = \underbrace{\vec{u} \times (\lambda\,\vec{u})}_{\vec{0}} = \underline{\vec{u} \times \mathtt{Q}\,\mathtt{X}}$$

§ 3 Vector product as a linear mapping It is interesting to see that for all $\vec{x}, \vec{y} \in \mathbb{R}^3$ there holds

$$\vec{x} \times \vec{y} = \begin{bmatrix} x_2\,y_3 - x_3\,y_2 \\ -x_1\,y_3 + x_3\,y_1 \\ x_1\,y_2 - x_2\,y_1 \end{bmatrix} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{1.51}$$

and thus we can introduce matrix

$$[\vec{x}]_\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{1.52}$$

rank $\begin{cases} 0 & \vec{x} = 0 \\ 2 & \vec{x} \neq 0 \end{cases}$

and write

$$\vec{x} \times \vec{y} = [\vec{x}]_\times\,\vec{y} \tag{1.53}$$

$$[\vec{x}]_\times^\top = -[\vec{x}]_\times$$

$$0 = \vec{u} \times (\lambda\,\vec{u}) = \vec{u} \times \mathtt{Q}\,\mathtt{X} = [\vec{u}]_\times\,\mathtt{Q}\,\mathtt{X}$$

$\underline{3\times 1}$ $\qquad$ $\underline{3\times 3 \quad 3\times 1}$

1.5 Operations with matrices & tensors

Matrices are a powerful tool which can be used in many ways. Here we review a few useful rules for matrix manipulation. The rules are often studied in multi-linear algebra and tensor calculus. We shall not review the theory of multi-linear algebra but will look at the rules from a phenomenological point of view. They are useful identities making an effective manipulation and concise notation possible.

§ 1 Kronecker product Let $\mathtt{A}$ be a $k \times l$ matrix and $\mathtt{B}$ be a $m \times n$ matrix

$$\mathtt{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1l} \\ a_{21} & a_{22} & \cdots & a_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kl} \end{bmatrix} \in \mathbb{R}^{k\times l} \quad \text{and} \quad \mathtt{B} \in \mathbb{R}^{m\times n} \tag{1.83}$$

then $k\,m \times l\,n$ matrix

$$\mathtt{C} = \mathtt{A} \otimes \mathtt{B} = \begin{bmatrix} a_{11}\,\mathtt{B} & a_{12}\,\mathtt{B} & \cdots & a_{1l}\,\mathtt{B} \\ a_{21}\,\mathtt{B} & a_{22}\,\mathtt{B} & \cdots & a_{2l}\,\mathtt{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}\,\mathtt{B} & a_{k2}\,\mathtt{B} & \cdots & a_{kl}\,\mathtt{B} \end{bmatrix} \tag{1.84}$$

is the matrix of the *Kronecker product of matrices* $\mathtt{A}$, $\mathtt{B}$ (in this order).

Notice that this product is associative, i.e. $(\mathtt{A} \otimes \mathtt{B}) \otimes \mathtt{C} = \mathtt{A} \otimes (\mathtt{B} \otimes \mathtt{C})$, but it is not commutative, i.e. $\mathtt{A} \otimes \mathtt{B} \neq \mathtt{B} \otimes \mathtt{A}$ in general. There holds a useful identity $(\mathtt{A} \otimes \mathtt{B})^\top = \mathtt{A}^\top \otimes \mathtt{B}^\top$.

§ 2 Matrix vectorization Let $\mathtt{A}$ be an $m \times n$ matrix

$$\mathtt{A} \quad = \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n} \tag{1.85}$$

$$v(\mathtt{A}) \quad = \quad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \\ a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \\ a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$m\,n \times 1$

We define operator $v(.)\colon \mathbb{R}^{m\times n} \to \mathbb{R}^{m\,n}$ which reshapes an $m \times n$ matrix $\mathtt{A}$ into a $m\,n \times 1$ matrix (i.e. into a vector) by stacking columns of $\mathtt{A}$ one above another

§3 From matrix equations to linear systems Kronecker product of matrices and matrix vectorization can be used to manipulate matrix equations in order to get systems of linear equations in the standard matrix form $\mathtt{A}\,\mathtt{x} = \mathtt{b}$. Consider, for instance, matrix equation

$$\mathtt{A}\,\mathtt{X}\,\mathtt{B} = \mathtt{C} \tag{1.91}$$

$\longrightarrow$ $x \longrightarrow Ax = c$

with matrices $\mathtt{A} \in \mathbb{R}^{m\times k}$, $\mathtt{X} \in \mathbb{R}^{k\times l}$, $\mathtt{B} \in \mathbb{R}^{l\times n}$, $\mathtt{C} \in \mathbb{R}^{m\times n}$. It can be verified by direct computation that

$$v(\mathtt{A}\,\mathtt{X}\,\mathtt{B}) = (\mathtt{B}^\top \otimes \mathtt{A})\, v(\mathtt{X}) \tag{1.92}$$

This is useful when matrices $\mathtt{A}$, $\mathtt{B}$ and $\mathtt{C}$ are known and we use Equation 1.91 to compute $\mathtt{X}$. Notice that matrix Equation 1.91 is actually equivalent to $m\,n$ scalar linear equations in $k\,l$ unknown elements of $\mathtt{X}$. Therefore, we should be able to write it in the standard form, e.g., as

$$\mathtt{M}\, v(\mathtt{X}) = v(\mathtt{C}) \tag{1.93}$$

with some $\mathtt{M} \in \mathbb{R}^{(m\,n)\times(k\,l)}$. We can use Equation 1.92 to get $\mathtt{M} = \mathtt{B}^\top \otimes \mathtt{A}$ which yields the linear system

$$v(\mathtt{A}\,\mathtt{X}\,\mathtt{B}) = v(\mathtt{C}) \tag{1.94}$$

$$(\mathtt{B}^\top \otimes \mathtt{A})\, v(\mathtt{X}) = v(\mathtt{C}) \tag{1.95}$$

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathtt{Q} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathtt{Q}\,\mathtt{X}$$

$$0 = \vec{u} \times (\lambda\, \vec{u}) = \vec{u} \times \mathtt{Q}\,\mathtt{X} = [\vec{u}]_\times\, \mathtt{Q}\,\mathtt{X}$$

$$[\vec{u}]_\times\, \mathtt{Q}\,\mathtt{X} = \mathbf{0}$$

$\vdots$

This gives three equations for each $\vec{u} \leftrightarrow \mathtt{X}$ correspondence. However, only two of them are linearly independet since $[\vec{u}]_\times$ has rank two. Now, we are in the position to employ Equation 1.95 which gives

$$
\begin{aligned}
[\vec{u}]_\times \, \mathtt{Q}\,\mathtt{X} &= \mathtt{0} \\
\mathtt{X}^\top \mathtt{Q}^\top \, [\vec{u}]_\times^\top &= \mathtt{0}^\top \\
v(\mathtt{X}^\top \mathtt{Q}^\top \, [\vec{u}]_\times^\top) &= v(\mathtt{0}^\top) \\
([\vec{u}]_\times \otimes \mathtt{X}^\top)\, v(\mathtt{Q}^\top) &= v(\mathtt{0}^\top) \\
\left(\begin{bmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{bmatrix} \otimes \mathtt{X}^\top \right) v(\mathtt{Q}^\top) &= v(\mathtt{0}^\top) \\
\begin{bmatrix} \mathtt{0}^\top & -w\,\mathtt{X}^\top & v\,\mathtt{X}^\top \\ w\,\mathtt{X}^\top & \mathtt{0}^\top & -u\,\mathtt{X}^\top \\ -v\,\mathtt{X}^\top & u\,\mathtt{X}^\top & \mathtt{0}^\top \end{bmatrix} v(\mathtt{Q}^\top) &= v(\mathtt{0}^\top)
\end{aligned}
$$

$$
\begin{aligned}
v(\mathtt{A}\,\mathtt{X}\,\mathtt{B}) &= v(\mathtt{C}) \\
(\mathtt{B}^\top \otimes \mathtt{A})\, v(\mathtt{X}) &= v(\mathtt{C})
\end{aligned}
$$

kron (A, B)

Easy to implement

For more correspondences numbered by i, we then get

$$
\begin{bmatrix}
\mathtt{0}^\top & -w_1\,\mathtt{X}_1^\top & v_1\,\mathtt{X}_1^\top \\
\mathtt{0}^\top & -w_2\,\mathtt{X}_2^\top & v_2\,\mathtt{X}_2^\top \\
 & \vdots & \\
w_1\,\mathtt{X}_1^\top & \mathtt{0}^\top & -u_1\,\mathtt{X}_1^\top \\
w_2\,\mathtt{X}_2^\top & \mathtt{0}^\top & -u_2\,\mathtt{X}_2^\top \\
 & \vdots & \\
-v_1\,\mathtt{X}_1^\top & u_1\,\mathtt{X}_1^\top & \mathtt{0}^\top \\
-v_2\,\mathtt{X}_2^\top & u_2\,\mathtt{X}_2^\top & \mathtt{0}^\top \\
 & \vdots &
\end{bmatrix} v(\mathtt{Q}^\top) = \mathtt{0} \qquad (5.40)
$$

$$
\underbrace{\begin{bmatrix} [\vec{u}_1]_\times \otimes X_1^\top \\ \vdots \\ [\vec{u}_6]_\times \otimes X_6^\top \end{bmatrix}}_{M_\otimes \in \mathbb{R}^{3 \cdot 6 \times 12}} q = 0
$$

Select rows $[1, 2, 4, 5, \ldots, 16]$ to get M'

which if, for $w = 1$, is equivalent to Equation 5.30. Notice that $v(\mathtt{Q}^\top) = \mathtt{q}$ from Equation 5.30.

Quiz

q23