Elements of Geometry for Computer Vision and Computer Graphics

Translation of Euclid's Elements by Adelardus Bathensis (1080–1152)

2021 Lecture 5

Tomas Pajdla

`pajdla@cvut.cz`

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the corresponding camera coordinate systems (C, β) with $\beta = [\vec{b}_1, \vec{b}_2, \vec{b}_3 = \overrightarrow{Co}]$ and (C, β') with $\beta' = [\vec{b}'_1, \vec{b}'_2, \vec{b}'_3 = \overrightarrow{Co'}]$.

Point X is projected to image points along the projection rays, which are intersected with π and π'. The projection of X in π is represented by vector $\vec{u}_\alpha = [u, v]^\top$. The projection of X in π' is represented by vector $\vec{u}'_{\alpha'} = [u', v']^\top$.

Vectors $\vec{x}$ and $\vec{x}'$ are two direction vectors of the same ray and hence *are linearly dependent*. Since they are both non-zero for $X \neq C$, their linear dependence is equivalent with

$$\exists \lambda \in \mathbb{R} : \lambda \vec{x}' = \vec{x} \tag{7.1}$$

To arrive at the relationship between the available coordinates of vectors $\vec{x}$ and $\vec{x}'$, we shall now pass from vectors to their coordinates. There holds

$$\lambda \vec{x}' = \vec{x} \tag{7.2}$$

$$\lambda \vec{x}'_{\beta'} = \vec{x}_{\beta'} \tag{7.3}$$

$$\lambda \vec{x}'_{\beta'} = \mathtt{H}\, \vec{x}_\beta \tag{7.4}$$

$\vec{x}_{\beta'} = H \vec{x}_\beta$

true for some 3×3 real matrix $\mathtt{H}$ with $\operatorname{rank} \mathtt{H} = 3$, which transforms coordinates of a vector from basis β to basis β'.

Considering the choices of camera coordinate systems, we see that

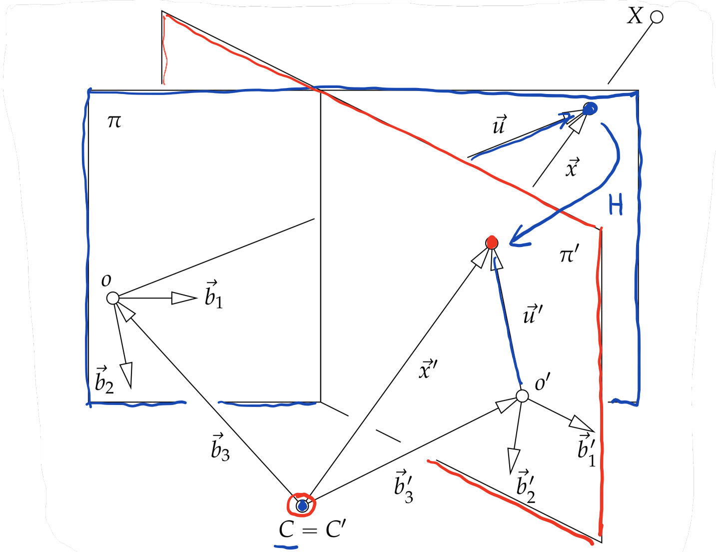
$$\boxed{\lambda \vec{x}'_{\beta'} = \mathtt{H}\, \vec{x}_\beta} \tag{7.5}$$

$$\vec{u}'_{\alpha'} \left\{ \quad \boxed{\lambda \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = \mathtt{H} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}} \quad \right\} \vec{u}_\alpha \tag{7.6}$$

We have obtained an interesting relationship. The above equations tell us that the image projections are related by a transformation, which depends only on image projections, and to find it, we do not need to know actual positions of points X in space. This is the consequence of having $C = C'$.

7.1 Homography between images with the same center

Let us consider two perspective cameras with identical projection centers $C = C'$, which project point X from space to their respective image planes π and π', Figure 7.1. We introduce image coordinate systems (o, α) with $\alpha = [\vec{b}_1, \vec{b}_2]$ in π and (o', α') with $\alpha' = [\vec{b}'_1, \vec{b}'_2]$ in π' and use them to construct



$$\tau \lambda \vec{x}'_{\beta'} = \tau H \vec{x}_\beta$$

$$\zeta \vec{x}'_{\beta'} = H_\zeta \vec{x}_\beta$$

$$H_\zeta = \tau H$$

§ 1 Relating homography matrix to camera projection matrix Matrix H is related to camera projection matrices. Consider two camera projections given by Equation 5.12

$$\zeta\, \vec{x}_\beta = \mathtt{P} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{K}\ \mathtt{R} \mid -\mathtt{K}\,\mathtt{R}\,\vec{C}_\delta \end{bmatrix} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = \mathtt{K}\,\mathtt{R}\ (\vec{X}_\delta - \vec{C}_\delta) \tag{7.7}$$

$$\zeta' \vec{x}'_{\beta'} = \mathtt{P}' \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{K}'\,\mathtt{R}' \mid -\mathtt{K}'\mathtt{R}'\vec{C}_\delta \end{bmatrix} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = \mathtt{K}'\,\mathtt{R}'\,(\vec{X}_\delta - \vec{C}_\delta) \tag{7.8}$$

$\mathtt{H} \overset{?}{\longleftrightarrow} \mathtt{P}, \mathtt{P}'$

$\mathtt{K}, \mathtt{R}, \mathtt{K}', \mathtt{R}'$ rank 3

3×3

for all $\vec{X}_\delta \in \mathbb{R}^3$, which gives

$$\zeta\, \mathtt{R}^\top \mathtt{K}^{-1}\, \vec{x}_\beta = \vec{X}_\delta - \vec{C}_\delta \tag{7.9}$$

$$\zeta'\, \mathtt{R}'^\top \mathtt{K}'^{-1} \vec{x}'_{\beta'} = \vec{X}_\delta - \vec{C}_\delta \tag{7.10}$$

$\forall\ \vec{X}_\delta \in \mathbb{R}^3$

and therefore

$$\zeta'\, \mathtt{R}'^\top \mathtt{K}'^{-1} \vec{x}'_{\beta'} = \zeta\, \mathtt{R}^\top\, \mathtt{K}^{-1}\, \vec{x}_\beta \tag{7.11}$$

$$\frac{\zeta'}{\zeta}\, \vec{x}'_{\beta'} = \mathtt{K}'\,\mathtt{R}'\,\mathtt{R}^\top \mathtt{K}^{-1}\, \vec{x}_\beta \tag{7.12}$$

$\mathtt{P}, \mathtt{P}'$ s.t. $\vec{C}_\delta = \vec{C}'_\delta$

$\downarrow$

$\mathtt{H}$

for all corresponding pairs of vectors $\vec{x}_\beta$, $\vec{x}'_{\beta'}$. Let us now compare Equation 7.12 with Equation 7.5, i.e. with

$$\lambda\, \vec{x}'_{\beta'} = \mathtt{H}\, \vec{x}_\beta \tag{7.13}$$

We see that

$$\mathtt{H} = \mathtt{K}'\,\mathtt{R}'\,\mathtt{R}^\top \mathtt{K}^{-1} \quad \text{when} \quad \lambda = \frac{\zeta'}{\zeta} \tag{7.14}$$

$\beta' \leftarrow \delta' \leftarrow ? \leftarrow \delta \leftarrow \beta$

Calibrated cameras

$K' = K = I$

$H = I R' R^T I = R' R^{-1}$ ... a rotation

$H = \xi R' R^{-1}$

$R \in \mathbb{R}^{3\times 3}$

$\underbrace{R^T R = I \qquad R^{-1} R = I}$

$|R| = 1$

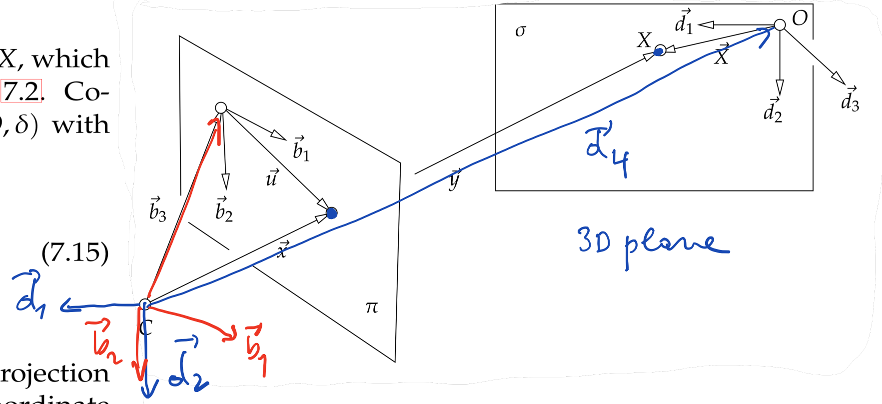
$|R| = -1$

7.2 Homography between two images of a plane

7.2.1 One image of a plane

Let study the relationship between the coordinates of 3D points X, which all lie in a plane σ, and their projections into an image, Figure 7.2. Coordinates of points X are measured in a coordinate system (O, δ) with $\delta = [\vec{d}_1, \vec{d}_2, \vec{d}_3]$. Vectors $\vec{d}_1, \vec{d}_2$ span plane σ and therefore

$$\vec{X}_\delta = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \tag{7.15}$$



2 projection planes π & σ

τ = $[\vec{d}_1, \vec{d}_2, \vec{d}_4]$

for some real x, y.

The points X are projected by a perspective camera with projection matrix $\mathtt{P}$ into image coordinates $\vec{u}_\alpha = [u, v]^\top$, w.r.t. an image coordinate system (o, α) with $\alpha = [\vec{b}_1, \vec{b}_2]$. The corresponding camera coordinate system is (C, β) with $\beta = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$.

To find the relationship between the coordinates of $\vec{X}_\delta$ and $\vec{u}_\alpha$, we project points X by $\mathtt{P}$ into projections $\vec{x}_\beta$ as

$$\zeta \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \zeta\, \vec{x}_\beta = \mathtt{P} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_3 & \mathtt{p}_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathtt{H}\, \vec{y}_\tau \tag{7.16}$$

where $\mathtt{p}_1, \mathtt{p}_2, \mathtt{p}_3, \mathtt{p}_4$ are the columns of $\mathtt{P}$.

Notice that 3×1 matrix $\vec{y}_\tau = [x, y, 1]^\top$ represents point X in the coordinate system (C, τ) with the basis $\boxed{\tau = (\vec{d}_1, \vec{d}_2, \vec{d}_4)}$, where the $\vec{d}_4 = \overrightarrow{CO}$ is the vector assigned to the pair of points (C, O). If point C is not in σ, then vectors $\vec{d}_1, \vec{d}_2, \vec{d}_4$ are independent and hence form a basis. Therefore, matrix

$$\boxed{\mathtt{H} = \begin{bmatrix} \mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_4 \end{bmatrix}} \tag{7.17}$$

represents a change of coordinates and has rank 3.

7.2.2 Two images of a plane

Let us consider a plane σ and two perspective cameras with (in general different) projection centers C and C', which do not lie in σ and the corresponding projection matrices $\mathtt{P}$ and $\mathtt{P}'$

$$\mathtt{P} = \begin{bmatrix} \mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_3 & \mathtt{p}_4 \end{bmatrix} \tag{7.18}$$

$$\mathtt{P}' = \begin{bmatrix} \mathtt{p}'_1 & \mathtt{p}'_2 & \mathtt{p}'_3 & \mathtt{p}'_4 \end{bmatrix} \tag{7.19}$$

where $\mathtt{p}_i \in \mathbb{R}^3$ and $\mathtt{p}'_i \in \mathbb{R}^3$, $i = 1, \ldots, 4$ stand for the columns of $\mathtt{P}$, $\mathtt{P}'$.

We establish coordinate systems (O, δ), (C, β), (C', β') in the standard way, see Figure 7.3 to get

$$\vec{X}_\delta = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \tag{7.20}$$

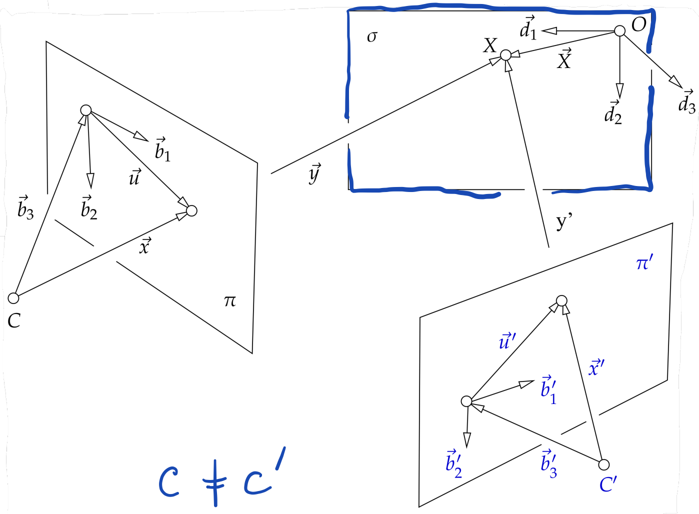
for some real x, y.

Point $X \in \sigma$ is projected to the cameras as

$$\zeta\, \vec{x}_\beta = \mathtt{P} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_3 & \mathtt{p}_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathtt{G}\, \vec{y}_\tau$$

$$\zeta'\, \vec{x}'_{\beta'} = \mathtt{P}' \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{p}'_1 & \mathtt{p}'_2 & \mathtt{p}'_3 & \mathtt{p}'_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{p}'_1 & \mathtt{p}'_2 & \mathtt{p}'_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathtt{G}'\, \vec{y}'_{\tau'}$$

$A\ X \in \sigma$



$C \neq C'$

for some $\zeta, \zeta' \in \mathbb{R}\backslash\{0\}$ and two new coordinate systems (C, τ) with $\tau = (\vec{d}_1, \vec{d}_2, \vec{d}_4)$, where the $\vec{d}_4 = \overrightarrow{CO}$ and (C', τ') with $\tau' = (\vec{d}_1, \vec{d}_2, \vec{d}'_4)$, where the $\vec{d}'_4 = \overrightarrow{CO'}$.

We see that there are two different vectors, $\vec{y}$ and $\vec{y}'$, which appear on the right hand side of the equations in different bases, i.e. as $\vec{y}_\tau$ and $\vec{y}'_{\tau'}$

$$\begin{aligned} \zeta\, \vec{x}_\beta &= \mathtt{G}\, \vec{y}_\tau \\ \zeta' \vec{x}'_{\beta'} &= \mathtt{G}' \vec{y}'_{\tau'} \end{aligned} \tag{7.22}$$

with $\mathtt{G} = [\mathtt{p}_1, \mathtt{p}_2, \mathtt{p}_4]$ and $\mathtt{G}' = [\mathtt{p}'_1, \mathtt{p}'_2, \mathtt{p}'_4]$.

Coordinate systems (C, τ) and (C', τ') are so special that

$$\vec{y}_\tau = \vec{y}'_{\tau'} \qquad \forall \vec{x}_\delta \in \mathbb{R}^3 \tag{7.23}$$

for all points in σ. Consider that

$$\vec{y}_\tau = (\vec{X} + \overrightarrow{CO})_\tau = \vec{X}_\tau + \vec{d}_{4\tau} = \vec{X}_{(\vec{d}_1, \vec{d}_2, \vec{d}_4)} + \vec{d}_{4(\vec{d}_1, \vec{d}_2, \vec{d}_4)} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.24}$$

$$\vec{y}'_{\tau'} = (\vec{X} + \overrightarrow{C'O})_{\tau'} = \vec{X}_{\tau'} + \vec{d}'_{4\tau'} = \vec{X}_{(\vec{d}_1, \vec{d}_2, \vec{d}'_4)} + \vec{d}'_{4(\vec{d}_1, \vec{d}_2, \vec{d}'_4)} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.25}$$

and therefore, when $C \notin \sigma$ and $C' \notin \sigma$, we get

$$\boxed{\zeta' \vec{x}'_{\beta'} = \mathtt{G}'\, \mathtt{G}^{-1} \zeta\, \vec{x}_\beta} \tag{7.26}$$

which we can write as

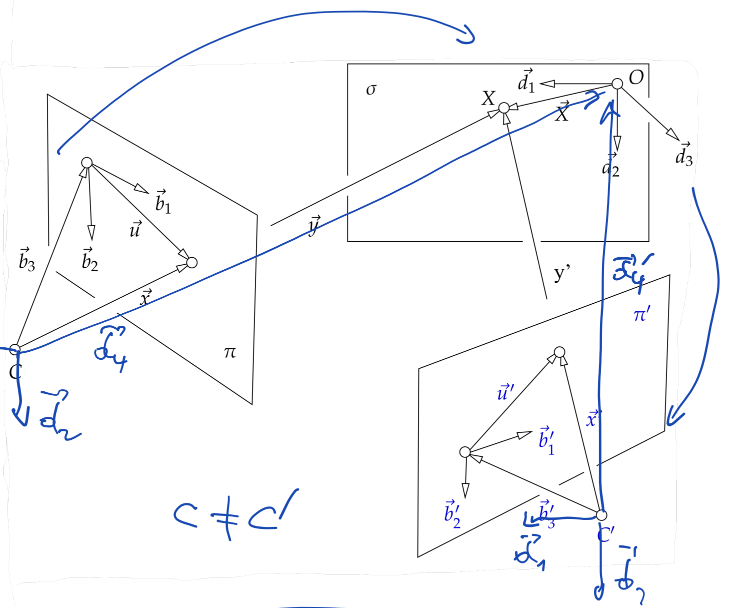
$$\boxed{\lambda\, \vec{x}'_{\beta'} = \mathtt{H}\, \vec{x}_\beta} \tag{7.27}$$

for $\lambda = \frac{\zeta'}{\zeta}$ and $\mathtt{H} = \mathtt{G}'\, \mathtt{G}^{-1}$. Clearly, $\mathtt{H} \in \mathbb{R}^{3\times 3}$, rank $\mathtt{H} = 3$.

Point $X \in \sigma$ is projected to the cameras as

$$\zeta\, \vec{x}_\beta = \mathtt{P} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = [\mathtt{p}_1 \;\; \mathtt{p}_2 \;\; \mathtt{p}_3 \;\; \mathtt{p}_4] \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = [\mathtt{p}_1 \;\; \mathtt{p}_2 \;\; \mathtt{p}_4] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathtt{G}\, \vec{y}_\tau$$

$$\zeta' \vec{x}'_{\beta'} = \mathtt{P}' \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = [\mathtt{p}'_1 \;\; \mathtt{p}'_2 \;\; \mathtt{p}'_3 \;\; \mathtt{p}'_4] \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = [\mathtt{p}'_1 \;\; \mathtt{p}'_2 \;\; \mathtt{p}'_4] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathtt{G}'\, \vec{y}'_{\tau'}$$



$$\begin{aligned} \mathtt{H} &= \mathtt{G}'\, \mathtt{G}^{-1} \\ &= [\mathtt{p}'_1 \;\; \mathtt{p}'_2 \;\; \mathtt{p}'_4]\, [\mathtt{p}_1 \;\; \mathtt{p}_2 \;\; \mathtt{p}_4]^{-1} \end{aligned}$$

7.2.3 Cameras with the same center

In the derivation of Equation 7.27, we have never asked for centers C, C' be different. Indeed, Equation 7.26 is perfetly valid even when $C = C'$. At the same time, however, there also holds Equation 7.14 true, and thus we have

$$\begin{aligned} \mathtt{H} &= \mathtt{G}'\,\mathtt{G}^{-1} && (7.28)\\ &= \begin{bmatrix}\mathtt{p}_1' & \mathtt{p}_2' & \mathtt{p}_4'\end{bmatrix}\begin{bmatrix}\mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_4\end{bmatrix}^{-1} && (7.29)\\ \mathtt{H} &= \mathtt{K}'\,\mathtt{R}'\,\mathtt{R}^\top\mathtt{K}^{-1} && (7.30)\\ &= \begin{bmatrix}\mathtt{p}_1' & \mathtt{p}_2' & \mathtt{p}_3'\end{bmatrix}\begin{bmatrix}\mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_3\end{bmatrix}^{-1} && (7.31)\end{aligned}$$

Let us see now purely algebraic argument why the above holds true. Since the cameras have the same projection center $\vec{C}_\delta = \begin{bmatrix}c_1 & c_2 & c_3\end{bmatrix}^\top$, we can write

$$\mathtt{p}_4 = -\mathtt{K}\,\mathtt{R}\,\vec{C}_\delta \quad \text{and} \quad \mathtt{p}_4' = -\mathtt{K}'\,\mathtt{R}'\,\vec{C}_\delta \tag{7.32}$$

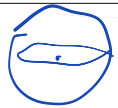
and hence

$$\begin{aligned} \mathtt{H} &= \mathtt{G}'\,\mathtt{G}^{-1} && (7.33)\\ &= \begin{bmatrix}\mathtt{p}_1' & \mathtt{p}_2' & \mathtt{p}_4'\end{bmatrix}\begin{bmatrix}\mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_4\end{bmatrix}^{-1} && (7.34)\\ &= \mathtt{K}'\,\mathtt{R}'\begin{bmatrix}\mathtt{i} & \mathtt{j} & -\vec{C}_\delta\end{bmatrix}\begin{bmatrix}\mathtt{i} & \mathtt{j} & -\vec{C}_\delta\end{bmatrix}^{-1}\mathtt{R}^\top\mathtt{K}^{-1} && (7.35)\\ &= \mathtt{K}'\,\mathtt{R}'\,\mathtt{R}^\top\mathtt{K}^{-1} && (7.36)\end{aligned}$$

with $\mathtt{i} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$ and $\mathtt{j} = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^\top$. We see that there always holds

$$\begin{bmatrix}\mathtt{p}_1' & \mathtt{p}_2' & \mathtt{p}_4'\end{bmatrix}\begin{bmatrix}\mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_4\end{bmatrix}^{-1} = \begin{bmatrix}\mathtt{p}_1' & \mathtt{p}_2' & \mathtt{p}_3'\end{bmatrix}\begin{bmatrix}\mathtt{p}_1 & \mathtt{p}_2 & \mathtt{p}_3\end{bmatrix}^{-1} \tag{7.37}$$

7.3 Spherical image



Consider a camera rotating around a center C and collecting n images all around such that every ray from C is captured in some image. We can choose one camera, e.g. the first one, and relate all other cameras to it as

$$\lambda_i \vec{x}_{\beta_1} \quad = \quad \mathtt{H}_i \vec{x}_{\beta_i}, \quad i = 1, \ldots, n \tag{7.38}$$

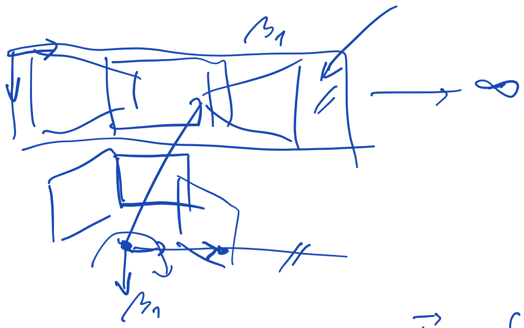
Since all vectors $\vec{x}$ were captured, there inevitably will appear a vector with coordinates

$$\vec{x}_{\beta_1} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \tag{7.39}$$

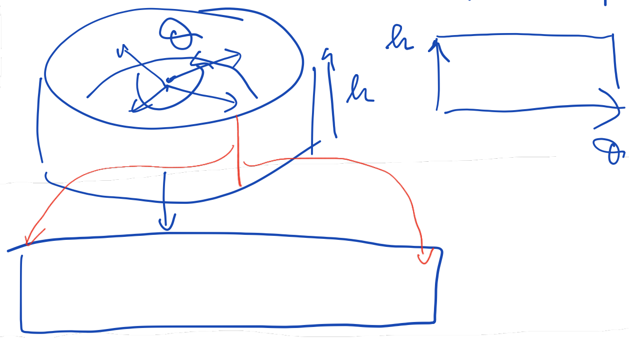
Such vector does not represent any point in the affine image plane π_1 of the first camera because it does not have the third coordinate equal to one. To be able to represent rays in all directions, we have to introduce *spherical image*, which is the set of all unit vectors in $\mathbb{R}^3$ (also called *omnidirectional image*). We sometimes use only a subset of the sphere, typically a cylinder, to capture *panoramic image*. In such a case, we can remap pixels onto such cylinder and then unwarp the cylinder into a plane. Notice however, that in such a representation, straight lines in space do not project to straight lines in images.

All equations we have developed so far work with minor modifications also for vectors with last zero coordinate. We will come back to it later when studying *projective plane* which is somewhere between the affine image plane and full spherical image.

Constructing a panorama



$\vec{x}_{\beta_1} = f(\theta, h)$



Cylindrical image

Quiz

then there holds

$$\exists\, \mathtt{H} \in \mathbb{R}^{3\times 3},\ \operatorname{rank} \mathtt{H} = 3,\ \text{so that}\ \forall\, [u,v]^\top \overset{corr}{\leftrightarrow} [u',v']^\top\ \exists \lambda \in \mathbb{R}:\ \lambda \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \mathtt{H} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{7.40}$$

true where $w = w' = 1$ for perspective images and may be general for spherical images.

In all three cases, coordinates of points are related by a homography. We have used linear algebra to derive the relationship between the coordinates of image points in the above form. The homography can be also represented in a different way.

To see that, we shall eliminate λ as follows

$$\lambda \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = \mathtt{H} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \tag{7.41}$$

$$\lambda u' = h_{11}\, u + h_{12}\, v + h_{13} \tag{7.42}$$

$$\lambda v' = h_{21}\, u + h_{22}\, v + h_{23} \tag{7.43}$$

$$\lambda 1 = h_{31}\, u + h_{32}\, v + h_{33} \tag{7.44}$$

$$u' = \frac{h_{11}\, u + h_{12}\, v + h_{13}}{h_{31}\, u + h_{32}\, v + h_{33}} \tag{7.45}$$

$$v' = \frac{h_{21}\, u + h_{22}\, v + h_{23}}{h_{31}\, u + h_{32}\, v + h_{33}} \tag{7.46}$$

We see that mapping h obtained as

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = h\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} \frac{h_{11}\, u + h_{12}\, v + h_{13}}{h_{31}\, u + h_{32}\, v + h_{33}} \\ \frac{h_{21}\, u + h_{22}\, v + h_{23}}{h_{31}\, u + h_{32}\, v + h_{33}} \end{bmatrix} \tag{7.47}$$

7.4 Homography – summary

Let us summarize the findings related to homography to see where it appears.

Let us encounter one of the following situations

1. **Two images with one projection center** Let $[u,v]^\top$ and $[u',v']^\top$ be coordinates of the projections of 3D points into two images by two perspective cameras with identical projection centers;
2. **Image of a plane**. Let $[u,v]^\top$ be coordinates of 3D points all in one plane σ, w.r.t. a oordinate system in σ and $[u',v']^\top$ coordinates of their projections by a perspective cameras with projection center not in the plane σ;
3. **Two images of a plane** Let $[u,v]^\top$ and $[u',v']^\top$ be coordinates of the projections of 3D points all in one plane σ, into two images by two perspective cameras with projection centers not in σ;

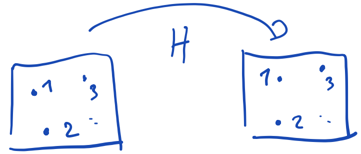
$$\vec{u}_\alpha \xrightarrow{\ h\ } \vec{u}'_{\alpha'}$$

non-linear

7.5 Computing homography from image matches

$$\mathtt{H} = \begin{bmatrix} \mathbf{h}_1^\top \\ \mathbf{h}_2^\top \\ \mathbf{h}_3^\top \end{bmatrix} \quad \text{and for the vector} \quad \mathbf{x} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \tag{7.56}$$

3x3



and rewrite the above matrix Equation 7.40 as

1) C = C'

3

$$\begin{aligned} \lambda\, u' &= \mathbf{h}_1^\top \mathbf{x} && (7.57)\\ \lambda\, v' &= \mathbf{h}_2^\top \mathbf{x} && (7.58)\\ \lambda &= \mathbf{h}_3^\top \mathbf{x} && (7.59) \end{aligned}$$

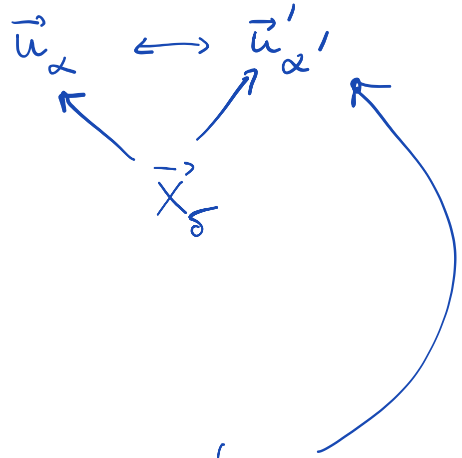
2) σ plane in 3D

Eliminate λ from the first two equations using the third one

2

$$\begin{aligned} (\mathbf{h}_3^\top \mathbf{x})\, u' &= \mathbf{h}_1^\top \mathbf{x} && (7.60)\\ (\mathbf{h}_3^\top \mathbf{x})\, v' &= \mathbf{h}_2^\top \mathbf{x} && (7.61)\\ & && (7.62) \end{aligned}$$

Image point correspondences



move all to the left hand side and reshape it using $\mathbf{x}^\top \mathbf{y} = \mathbf{y}^\top \mathbf{x}$

$$\begin{aligned} \mathbf{x}^\top \mathbf{h}_1 - (u'\mathbf{x}^\top)\, \mathbf{h}_3 &= 0 && (7.63)\\ \mathbf{x}^\top \mathbf{h}_2 - (v'\mathbf{x}^\top)\, \mathbf{h}_3 &= 0 && (7.64)\\ & && (7.65) \end{aligned}$$

Introduce notation

9x1

$$\mathbf{h} = \begin{bmatrix} \mathbf{h}_1^\top & \mathbf{h}_2^\top & \mathbf{h}_3^\top \end{bmatrix}^\top \tag{7.66}$$

and express the above two equations in a matrix form

$$\boxed{\begin{bmatrix} u & v & 1 & 0 & 0 & 0 & -u'u & -u'v & -u' \\ 0 & 0 & 0 & u & v & 1 & -v'u & -v'v & -v' \end{bmatrix} \mathbf{h} = 0} \tag{7.67}$$

1 correspondence

x' ↔ x

2 eqns.

Every correspondence $\boxed{[u,v]^\top \overset{corr}{\leftrightarrow} [u',v']^\top}$ brings two rows to a matrix

$$\begin{bmatrix} u & v & 1 & 0 & 0 & 0 & -u'u & -u'v & -u' \\ 0 & 0 & 0 & u & v & 1 & -v'u & -v'v & -v' \\ & & & & \vdots & & & & \end{bmatrix} \mathbf{h} = 0 \tag{7.68}$$

$$\mathtt{M} \quad \mathbf{h} = 0 \tag{7.69}$$

How many corresp.?

$M \in \mathbb{R}^{2n \times 9}$

$h \neq 0$

$\longrightarrow H \neq 0$ (rank $H = 3$)

1D space of h vectors

when $n = 4$: $M \in \mathbb{R}^{8\times 9}$

rank $M = 8 \Rightarrow$ 1D s. of. sols.

If $\xi\, \mathtt{G} = \mathtt{H}$, $\xi \neq 0$ then both $\mathtt{G}, \mathtt{H}$ represent the same homography. We are therefore looking for one-dimensional subspaces of 3×3 matrices of rank 3. Each such subspace determines one homography. Also note that the zero matrix, $\mathtt{0}$, does not represent an interesting mapping.

We need therefore at least 4 correspondences in a general position to obtain rank 8 matrix $\mathtt{M}$. By a general position we mean that the matrix $\mathtt{M}$ must have *rank* 8 to provide a single one-dimensional subspace of its solutions. This happens when no 3 out of the 4 points are on the same line.

Notice that $\mathtt{M}$ can be written in the form

$$\mathtt{M} = \begin{bmatrix} u_1 & v_1 & 1 & 0 & 0 & 0 & -u_1'u_1 & -u_1'v_1 & -u_1' \\ u_2 & v_2 & 1 & 0 & 0 & 0 & -u_2'u_2 & -u_2'v_2 & -u_2' \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & u_1 & v_1 & 1 & -v_1'u_1 & -v_1'v_1 & -v_1' \\ 0 & 0 & 0 & u_2 & v_2 & 1 & -v_2'u_2 & -v_2'v_2 & -v_2' \\ & & & & \vdots & & & & \end{bmatrix} \tag{7.70}$$

$$\mathtt{M} = \begin{bmatrix} \mathbf{x}_1^\top & \mathbf{0}^\top & -u_1'\,\mathbf{x}_1^\top \\ \mathbf{x}_2^\top & \mathbf{0}^\top & -u_2'\,\mathbf{x}_2^\top \\ & \vdots & \\ \mathbf{0}^\top & \mathbf{x}_1^\top & -v_1'\,\mathbf{x}_1^\top \\ \mathbf{0}^\top & \mathbf{x}_2^\top & -v_2'\,\mathbf{x}_2^\top \\ & \vdots & \end{bmatrix}$$

7.5.2 Advanced procedure for computing a general $\mathtt{H}$

with $\mathbf{x} = [u, v, w]^\top$ and $\mathbf{x}' = [u', v', w']^\top$ and follow the derivation in § 1 to get

Tensorial framework

Eliminate λ

$[x']_\times \lambda x' = 0$

$x' \times (\lambda x') = 0$

rank = 2

$$\lambda\,\mathbf{x}' = \mathtt{H}\,\mathbf{x} \tag{7.73}$$

$$[\mathbf{x}']_\times\,\mathtt{H}\,\mathbf{x} = \mathbb{0} \tag{7.74}$$

$$\mathbf{x}^\top \mathtt{H}^\top\,[\mathbf{x}']_\times^\top = \mathbb{0}^\top \tag{7.75}$$

$$v(\mathbf{x}^\top \mathtt{H}^\top\,[\mathbf{x}']_\times^\top) = v(\mathbb{0}^\top) \tag{7.76}$$

$$\boxed{([\mathbf{x}']_\times \otimes \mathbf{x}^\top)\,v(\mathtt{H}^\top) = v(\mathbb{0}^\top)} \tag{7.77}$$

Kronecker product

3

$$\left(\begin{bmatrix} 0 & -w' & v' \\ w' & 0 & -u' \\ -v' & u' & 0 \end{bmatrix} \otimes \mathbf{x}^\top\right) v(\mathtt{H}^\top) = v(\mathbb{0}^\top) \tag{7.78}$$

$$\begin{bmatrix} \mathbb{0}^\top & -w'\mathbf{x}^\top & v'\mathbf{x}^\top \\ w'\mathbf{x}^\top & \mathbb{0}^\top & -u'\mathbf{x}^\top \\ -v'\mathbf{x}^\top & u'\mathbf{x}^\top & \mathbb{0}^\top \end{bmatrix} v(\mathtt{H}^\top) = v(\mathbb{0}^\top) \tag{7.79}$$

→ 1 corresp → 3 equs

2 indep

For more correspondences numbered by i, we then get

$$\begin{bmatrix} \mathbb{0}^\top & -w_1'\mathbf{x}_1^\top & v_1'\mathbf{x}_1^\top \\ \mathbb{0}^\top & -w_2'\mathbf{x}_2^\top & v_2'\mathbf{x}_2^\top \\ & \vdots & \\ w_1'\mathbf{x}_1^\top & \mathbb{0}^\top & -u_1'\mathbf{x}_1^\top \\ w_2'\mathbf{x}_2^\top & \mathbb{0}^\top & -u_2'\mathbf{x}_2^\top \\ & \vdots & \\ -v_1'\mathbf{x}_1^\top & u_1'\mathbf{x}_1^\top & \mathbb{0}^\top \\ -v_2'\mathbf{x}_2^\top & u_2'\mathbf{x}_2^\top & \mathbb{0}^\top \\ & \vdots & \end{bmatrix} v(\mathtt{H}^\top) = \mathbb{0} \tag{7.80}$$

$M\, v(H^T) = 0$

$M \cdot h = 0$

$\in \mathbb{R}^{3\cdot 4 \times 9}$

$\mathbb{R}^{12 \times 9}$ rank = 8

which is, for $w = 1$, equivalent to Equation 5.30. Notice that $v(\mathtt{H}^\top) = \mathbf{h}$ from Equation 7.69.

Quiz