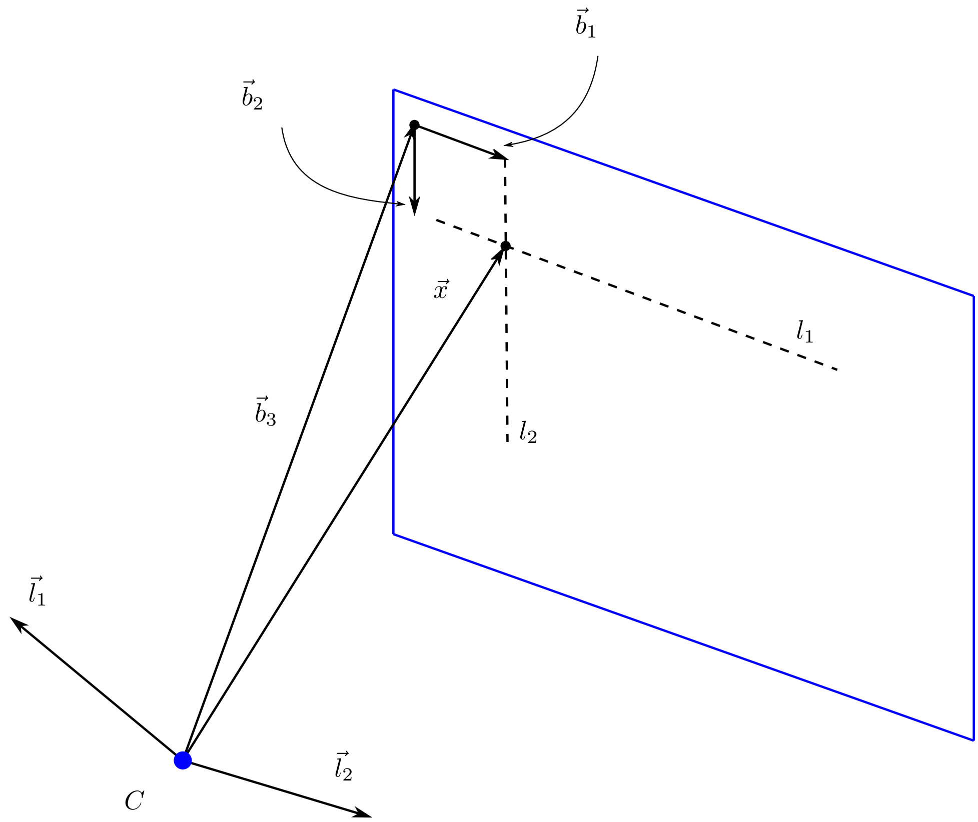
GVG'2022 Lab-08 Solution

Task 1. *Let us have two lines in the image* l_1 *and* l_2 *given by:*

$$l_1: \ u = 1, \quad l_2: \ v = 1.$$

Find their intersection in $\mathbb{A}^2$*, if exists (using techniques of projective geometry).*



Solution: Obviously, it is not necessary to use the techniques of projective geometry (the cross product rule for the intersection of 2 lines): we could simply setup the system of linear equations

$$\begin{cases} 1 \cdot u + 0 \cdot v = 1 \\ 0 \cdot u + 1 \cdot v = 1 \end{cases} \iff \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathtt{M}} \begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{b}}$$

and solve it by $\mathtt{M}^{-1}\mathbf{b}$. However, the above system will not have any solutions if the lines are parallel and not identical ($\mathbf{b}$ will not belong to rng $\mathtt{M}$).

We can also rewrite the above system as

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}}_{\mathtt{A}} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and use Gaussian elimination to solve it. If the kernel of $\mathtt{A}$ has the generator with the last coordinate zero, then the system has no solutions.

Another way to find the kernel of $\mathtt{A}$ is to compute the cross product of the 2 rows of $\mathtt{A}$. The result may be interpreted as the intersection of l_1 and l_2 in $\mathbb{P}^2$, since now 3×1 vectors of numbers with the last coordinate zero represent points at infinity of $\mathbb{P}^2$. (We see that it is easier to work in $\mathbb{P}^2$ rather than in $\mathbb{A}^2$ since we don't need to distinguish the 2 cases of parallel and not parallel lines.) The homogeneous representatives of l_1 and l_2 in $\mathbb{P}^2$ are:

$$\mathbf{l}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{l}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Their cross product is

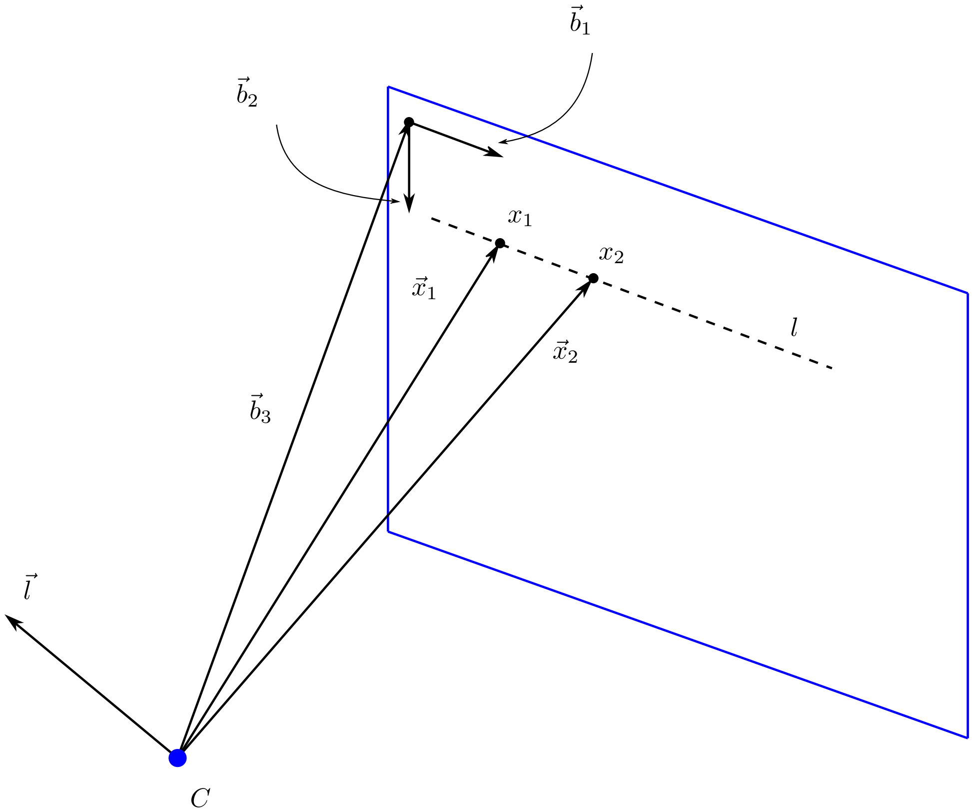
$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We see that the last coordinate is nonzero, which means that l_1 and l_2 intersect in $\mathbb{A}^2$. (If it was zero, then they would be parallel and wouldn't intersect in $\mathbb{A}^2$, but in $\mathbb{P}^2$.) To find the point of intersection in $\mathbb{A}^2$ we need to find the representative of $[\mathbf{x}]$ with the last coordinate 1 and take the first 2 coordinates, which are $\begin{bmatrix} 1 & 1 \end{bmatrix}^\top$. □

Task 2. *Let us have two image points x_1 and x_2 defined by*

$$\vec{u}_{1\alpha} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{u}_{2\alpha} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Find the line in the image (in the form $au + bv + c = 0$) passing through them (using techniques of projective geometry).



Solution: Again, it is not necessary to use techniques of projective geometry (the cross product rule for the line passing through 2 points): we could simply setup the system of linear equations

$$\begin{cases} a \cdot 1 + b \cdot 1 + c = 0 \\ a \cdot 2 + b \cdot 1 + c = 0 \end{cases} \iff \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}}_{\mathtt{M}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and solve it by Gaussian elimination of $\mathtt{M}$.

The homogeneous representatives of x_1 and x_2 are

$$\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}2\\1\\1\end{bmatrix}$$

Their cross product is

$$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix} \times \begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

Passing to affine representation of the line given by $\mathbf{l}$ we get

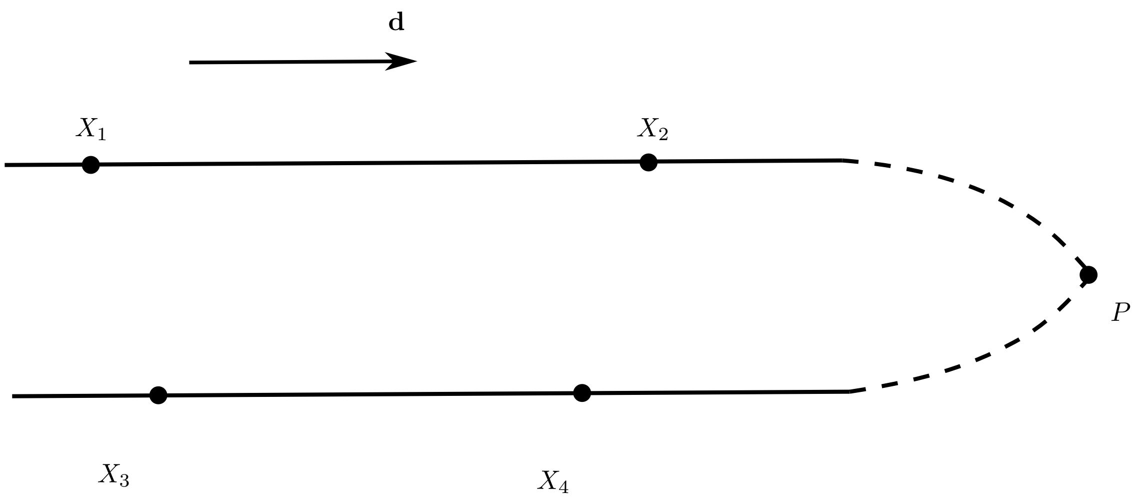
$$l:\ 0 \cdot u + 1 \cdot v - 1 = 0 \Rightarrow l:\ v = 1.$$

The point at infinity of $\mathbb{P}^2$ represented by $\begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$ associated to l doesn't belong to l, but to its projective closure $\bar{l}$. □

Task 3. *Let us have two lines L_1 and L_2 in $\mathbb{A}^3$ given by:*

$$L_1:\ \vec{X}_{1\delta} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \vec{X}_{2\delta} = \begin{bmatrix}1\\0\\1\end{bmatrix}; \quad L_2:\ \vec{X}_{3\delta} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \vec{X}_{4\delta} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Find the intersection of L_1 and L_2 (if exists) in the projective space $\mathbb{P}^3$.



Solution: While every two lines in $\mathbb{P}^2$ intersect, this is not the case in $\mathbb{P}^3$. Obviously, two lines L_1 and L_2 defined by 4 points from $\mathbb{A}^3$ intersect in $\mathbb{P}^3$ if and only if these lines lie in a plane. Algebraically, this condition can be expressed as

$$\det\left(\begin{bmatrix} \vec{X}_{1\delta}^\top & 1 \\ \vec{X}_{2\delta}^\top & 1 \\ \vec{X}_{3\delta}^\top & 1 \\ \vec{X}_{4\delta}^\top & 1 \end{bmatrix}\right) = 0.$$

Notice that the determinant of that matrix is zero if and only if it has a nontrivial kernel, whose generator (or generators if the 4 points are degenerate) defines the coefficients of the plane. For this task we can see that

$$\det\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right) = 0$$

since the first and the last column are equal (meaning the 4 columns are linearly dependent). This means that the lines L_1 and L_2 intersect in $\mathbb{P}^3$ (however, they may not intersect in $\mathbb{A}^3$, which happens when they are parallel).

In order to find the intersection of L_1 and L_2 in $\mathbb{P}^3$ we need to construct a 3×4 matrix, whose rows represent 3 different planes: one passes through L_1, the second – through L_2, and the third contains them both. They can be constructed as follows:

$$\vec{X}_{1\delta} \times \vec{X}_{2\delta} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \sigma_1 : \ 0 \cdot x + (-1) \cdot y + 0 \cdot z + 0 = 0 \Rightarrow \boldsymbol{\sigma}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{X}_{3\delta} \times \vec{X}_{4\delta} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \sigma_2 : \ 1 \cdot x + (-1) \cdot y + 0 \cdot z + 0 = 0 \Rightarrow \boldsymbol{\sigma}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\ker\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right) = \left\langle \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle \Rightarrow \boldsymbol{\sigma}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

To find the intersection of L_1 and L_2 in $\mathbb{P}^3$ we need to find the kernel of the following matrix:

$$\begin{bmatrix} \boldsymbol{\sigma}_1^\top \\ \boldsymbol{\sigma}_2^\top \\ \boldsymbol{\sigma}_3^\top \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\ker\left(\begin{bmatrix} \boldsymbol{\sigma}_1^\top \\ \boldsymbol{\sigma}_2^\top \\ \boldsymbol{\sigma}_3^\top \end{bmatrix}\right) = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle \Rightarrow P = \overline{L_1} \cap \overline{L_2} = \left[\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right] \in \mathbb{P}^3$$

Notice that the intersection of the projective closures of L_1 and L_2 is a point at infinity of $\mathbb{P}^3$ since L_1 and L_2 are parallel. Also notice that the first 3 coordinates $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$ of P define the direction vector $\mathbf{d}$ of the given lines. □

Task 4. *Let the camera be given by the following camera projection matrix*

$$\mathtt{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Let the rectangle in space be defined by the following 4 points:

$$\vec{X}_{1\delta} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{X}_{2\delta} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{X}_{3\delta} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{X}_{4\delta} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Find the horizon of the plane defined by the rectangle.

Solution: In order to draw a picture what is happening in this task we need to take one of the cameras which has the given camera projection matrix.

Remark. *Since the set of camera projection matrices is in bijective correspondence with the set of triples* $(\mathtt{K}, \mathtt{R}, \vec{C}_\delta)$ *and every camera is uniquely defined by a 4-tuple* $(f, \mathtt{K}, \mathtt{R}, \vec{C}_\delta)$*, then it is obvious that for fixed* $\mathtt{K}, \mathtt{R}$ *and* $\vec{C}_\delta$ *the set of cameras*

$$\left\{ (f, \mathtt{K}, \mathtt{R}, \vec{C}_\delta) \;\middle|\; f \in \mathbb{R}^+ \right\}$$

have the same camera projection matrix. Out of those we take the one with $f = 1$ *to create a picture for this task (see Figure 1). Since* $\mathtt{P}_{1:3,1:3} = \mathtt{I}$ *and* $\mathtt{P}_{1:3,4} = \mathbf{0}$ *results into* $\mathtt{K} = \mathtt{R} = \mathtt{I}$ *and* $\vec{C}_\delta = \mathbf{0}$*, then*

$$O = C, \quad \mathtt{T}_{\delta \to \gamma} = \frac{1}{f}\mathtt{R} = \mathtt{I}, \quad \mathtt{T}_{\gamma \to \beta} = \mathtt{K} = \mathtt{I},$$

which means that $(O, \delta) = (C, \gamma) = (C, \beta)$.

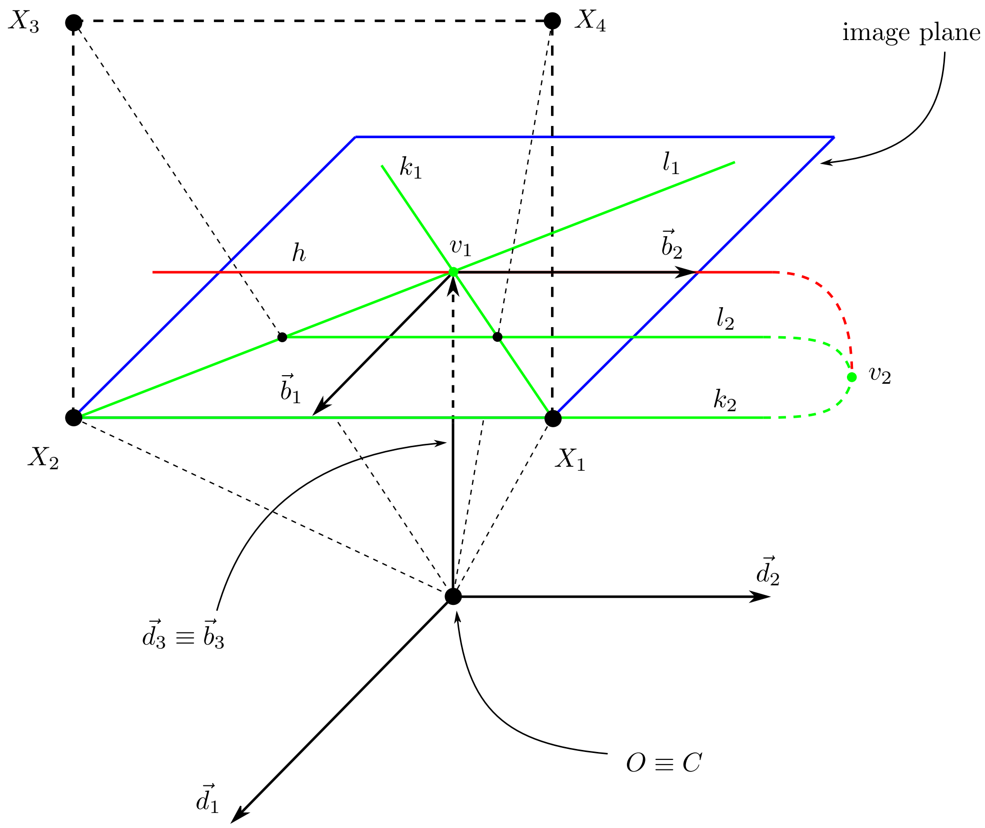


Figure 1: Camera observes the square $\square X_1X_2X_3X_4$

We first project the world points to the camera:

$$\mathbf{x}_1 = \mathtt{P}\begin{bmatrix}\vec{X}_{1\delta}\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \mathbf{x}_2 = \mathtt{P}\begin{bmatrix}\vec{X}_{2\delta}\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix}, \quad \mathbf{x}_3 = \mathtt{P}\begin{bmatrix}\vec{X}_{3\delta}\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\2\end{bmatrix}, \quad \mathbf{x}_4 = \mathtt{P}\begin{bmatrix}\vec{X}_{4\delta}\\1\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix}.$$

There is no need to find the image points $\vec{u}_{i\alpha}, i = 1, \ldots, 4$ (by dividing $\mathbf{x}_i$ by the last coordinate), since it is easier to work in homogeneous coordinates to work with lines in $\mathbb{P}^2$ and their intersections. In order to find the horizon (the projection of a line at infinity of the plane τ defined by $\square X_1X_2X_3X_4$) it is sufficient to find two vanishing points of two pairs of parallel lines from τ. Those 2 pairs will be defined by $(\overline{X_2X_3}, \overline{X_1X_4})$ and $(\overline{X_1X_2}, \overline{X_3X_4})$.

Two find the representatives of the images k_1 and l_1 of the first pair we apply the cross product rule to the homogeneous representatives of the projected points to the camera:

$$\mathbf{k}_1 = \mathbf{x}_1 \times \mathbf{x}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{l}_1 = \mathbf{x}_2 \times \mathbf{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

The vanishing point associated to the pair of world lines $(\overline{X_2X_3}, \overline{X_1X_4})$ is then the intersection of k_1 and l_1. In homogeneous coordinates we have:

$$\mathbf{v}_1 = \mathbf{k}_1 \times \mathbf{l}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

Since the last coordinate of $\mathbf{v}_1$ is nonzero, the we can pass to the affine coordinates (by dividing by the last coordinate and taking the first two) and see that $v_{1_{(o,\alpha)}} = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$.

Similarly, the representatives of the images k_2 and l_2 of the second pair of world lines $(\overline{X_1X_2}, \overline{X_3X_4})$ are:

$$\mathbf{k}_2 = \mathbf{x}_1 \times \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{l}_2 = \mathbf{x}_3 \times \mathbf{x}_4 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$$

The vanishing point associated to the pair of world lines $(\overline{X_1X_2}, \overline{X_3X_4})$ is then the intersection of k_2 and l_2 (in $\mathbb{P}^2$). In homogeneous coordinates we have:

$$\mathbf{v}_2 = \mathbf{k}_2 \times \mathbf{l}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \times \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

Since the last coordinate of $\mathbf{v}_2$ is zero, then v_2 is a point at infinity of $\mathbb{P}^2$. (This is logical since the world lines $\overline{X_1X_2}$ and $\overline{X_3X_4}$ are parallel to the image plane of the camera).

To find the horizon we need to find the line passing through the points v_1 (visible in the image) and v_2 (not visible in the image). We do this again by the cross product:

$$\mathbf{h} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Passing to affine coordinates, we see that

$$h = \{(u, v) \in \mathbb{R}^2 \mid 1 \cdot u + 0 \cdot v + 0 = 0\}$$

is a line $u = 0$ in the image. Notice that the horizon of τ is (always) the intersection of the plane parallel to τ and passing through the camera center C and the image plane of the camera. □