GVG Lab-13 Solution

Task 1. *1. Find the unknowns a, b, c in the following fundamental matrix*

$$\mathtt{F} = \begin{bmatrix} a & 1 & 1 \\ b & 1 & 0 \\ c & 2 & 1 \end{bmatrix}$$

when the epipole in the first image is $[1,1]^\top$.

2. Find the epipolar line in the second image that corresponds to point $[0,1]^\top$ in the first image.

Solution:

1. The epipole $\vec{e}_{1\beta_1}$ in the first image generates the kernel of F:

$$\mathtt{F}\vec{e}_{1\beta_1} = \mathbf{0}$$

$$\begin{bmatrix} a & 1 & 1 \\ b & 1 & 0 \\ c & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff a = -2, b = -1, c = -3.$$

2. The epipolar line in the second image is given by

$$\mathbf{l} = \mathtt{F}\vec{x}_{1\beta_1} = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

□

Task 2. *Consider two cameras with camera projection matrices*

$$\mathtt{P}_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathtt{P}_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Find point $\vec{X}_\delta$ in space that projects into image points $\vec{u}_{1\alpha_1} = [2,1]^T$, $\vec{u}_{2\alpha_2} = [2,0]^T$.

Solution: We can write

$$\zeta_1 \vec{x}_{1\beta_1} = \mathtt{P}_1 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \zeta_2 \vec{x}_{2\beta_2} = \mathtt{P}_2 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}$$

Substituting the known values from the task and letting $\vec{X}_\delta = \begin{bmatrix} x & y & z \end{bmatrix}^\top$ we obtain

$$\zeta_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \quad \zeta_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{cases} 2\zeta_1 = x + 1 \\ \zeta_1 = y \\ \zeta_1 = z \end{cases} \qquad \begin{cases} 2\zeta_2 = z + 1 \\ 0 = -y + 1 \\ \zeta_2 = x \end{cases}$$

Eliminating ζ_1 and ζ_2 we get

$$\begin{cases} 2z = x + 1 \\ z = y \\ 2x = z + 1 \\ 0 = -y + 1 \end{cases}$$

$$x = y = z = 1 \Rightarrow \zeta_1 = z = 1, \zeta_2 = x = 1.$$

The world point $\vec{X}_\delta$ could be also computed without eliminating ζ_1 and ζ_2 by computing the kernel of 6×6 matrix:

$$\begin{bmatrix} \vec{x}_{1\beta_1} & \mathbf{0} & -\mathtt{P}_1 \\ \mathbf{0} & \vec{x}_{2\beta_2} & -\mathtt{P}_2 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vec{X}_\delta \\ 1 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{0}$$

We see that the kernel is generated by vector $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top$. □

Task 3. *Suppose we are given the essential matrix*

$$\mathtt{G} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Compute 4 pairs $(\mathtt{R}, \vec{t}_\delta)$ *with* $\|\vec{t}_\delta\| = 1$ *such that they define* $\mathtt{G}$ *(up to scale), i.e.*

$$\mathtt{G} = \lambda \mathtt{R} \left[\vec{t}_\delta\right]_\times$$

for some nonzero $\lambda \in \mathbb{R}$*.*

Solution:

Remark. *In practice, using only image measurements we cannot recover the physical solution* $(\mathtt{R}, \vec{C}_\delta)$ *which generated the images as well as the physical essential matrix* $\mathtt{E} = \mathtt{R}\left[\vec{C}_\delta\right]_\times$*. The reason for this is that after fixing the center and the rotation of the world coordinate frame in the first camera ([2], Equation (12.44)) there is still a scaling symmetry present in the equations*

$$\zeta_1 \vec{x}_{1\gamma_1} = \left[\mathtt{I} \mid \mathbf{0}\right] \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \zeta_2 \vec{x}_{2\gamma_2} = \left[\mathtt{R} \mid -\mathtt{R}\vec{C}_\delta\right] \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}$$

given by

$$\begin{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vec{X}_\delta \\ \vec{C}_\delta \end{bmatrix} \\ \mathtt{R} \end{bmatrix} \mapsto \begin{bmatrix} \lambda \cdot \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vec{X}_\delta \\ \vec{C}_\delta \end{bmatrix} \\ \mathtt{R} \end{bmatrix}$$

for a nonzero $\lambda \in \mathbb{R}$ *(the unknowns are depicted in red).* ***In other words, we can scale the basic vectors of the world coordinate system*** δ ***by*** λ ***and it will not change the image points*** $\vec{x}_{1\gamma_1}$ ***and*** $\vec{x}_{2\gamma_2}$***. This symmetry corresponds to choosing a measuring unit (e.g. millimeters, meters, inches, etc.) to represent the world point*** X ***and the camera center*** C***.*** *This scaling symmetry obviously remains after reformulating the problem in terms of the fundamental matrix* $\mathtt{F}$ *(7,8 point problems) or in terms of the essential matrix* $\mathtt{E}$ *(5 point problem). By dehomogenizing the formulations (e.g. fixing* β *to* 1 *in [2, Equation (12.34)] or adding the last equation in [2, Equation (12.109)]) we fix the measuring unit.* ***An important remark to be made here is that we don't know the relation of the measuring unit that we fixed to the known measuring units, like millimeters, meters, inches, etc. However, it doesn't matter in which measuring unit we represent the 3D reconstruction of the scene (and it doesn't matter that we don't know the relation to the known units), because all the units are equivalent. (We just won't be able to imagine how big is the reconstructed scene). While we can still use the***

reconstructed scene for visual localization, this approach of 3D reconstruction cannot be used for obstacle avoidance, e.g., in drones (since you don't know how far the obstacle is in known units). That's why, for obstacle avoidance, stereo cameras are used (where you know the distance $\vec{C}_\delta$ between the lenses in known measuring units).

First we compute the Frobenius norm of $\mathtt{G}$ ([2, Equation (12.50)])

$$\|\mathtt{G}\|_F = \sqrt{\sum_{i,j=1}^{3} \mathtt{G}_{ij}^2} = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}.$$

The normalized matrix $\overline{\mathtt{G}}$ ([2, Equation (12.53)]) then looks like

$$\overline{\mathtt{G}} = \frac{\sqrt{2}\mathtt{G}}{\|\mathtt{G}\|_F} = \frac{\sqrt{2}}{2\sqrt{2}}\mathtt{G} = \frac{1}{2}\mathtt{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We now know that $\overline{\mathtt{G}}$ is representable by $s_1 \mathtt{R} \left[\vec{t_\delta}\right]_\times$ for some $s_1 \in \{+1,-1\}$ ([2, Equation (12.53)]), where $\vec{t_\delta} = \frac{\vec{C_\delta}}{\|\vec{C_\delta}\|}$. Notice that $\vec{t_\delta}$ belongs to the kernel of $\overline{\mathtt{G}}$:

$$\overline{\mathtt{G}}\vec{t_\delta} = s_1 \mathtt{R} \underbrace{\left[\vec{t_\delta}\right]_\times \vec{t_\delta}}_{\mathbf{0}} = \mathbf{0}.$$

However, since the kernel is 1-dimensional, it has 2 representatives of the unit norm (with opposite signs), and clearly, knowing only $\overline{\mathtt{G}}$, we don't know which of them corresponds to $\vec{t_\delta}$. Thus, there are 2 possibilities for $\vec{t_\delta}$. We compute the kernel of $\overline{\mathtt{G}}$:

$$\overline{\mathtt{G}}\mathbf{v} = \mathbf{0}$$

and get the generator

$$\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \vec{t}_{+\delta} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{t}_{-\delta} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

1) $\vec{t}_{+\delta} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We know that there holds

$$\overline{\mathtt{G}} = s_1 \mathtt{R} \left[\vec{t}_{+\delta}\right]_\times$$

for some $s_1 \in \{+1,-1\}$. We substitute $\overline{\mathtt{G}}$ and $\vec{t}_{+\delta}$ to this equation and get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = s_1 \mathtt{R} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, there are 2 possibilities for s_1.

1.1) $s_1 = 1$. We get the following equation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathtt{R}_+ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

We use the well-known fact from linear algebra: if $\mathtt{R}$ is a 3×3 rotation matrix and $\mathbf{x}$ and $\mathbf{y}$ are linearly independent vectors from $\mathbb{R}^{3\times 1}$, then $(\mathtt{R}\mathbf{x}) \times (\mathtt{R}\mathbf{y}) = \mathtt{R}(\mathbf{x} \times \mathbf{y})$ ([2, Equation (12.59)]). Looking at Equation (1) we see that we can take $\mathbf{x}$ and $\mathbf{y}$ to be the 1st and the 2nd columns of $\left[\vec{t}_{+\delta}\right]_\times$, since they are linearly independent. Thus,

$$\left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right] = \mathtt{R}_+ \left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right]$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \mathtt{R}_+ \begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}$$

Now, unlike in Equation (1), we see that the matrix to the right of $\mathtt{R}_+$ is invertible, and we can compute $\mathtt{R}_+$ uniquely by

$$\mathtt{R}_+ = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}^{-1} = \begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}$$

1.2) $s_1 = -1$. We get

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} = \mathtt{R}_- \begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix} \tag{2}$$

We apply the same trick with the cross product and get

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \mathtt{R}_- \begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}$$

$$\mathtt{R}_- = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}^{-1} = \begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}$$

2) $\vec{t}_{-\delta} = \begin{bmatrix}0\\0\\-1\end{bmatrix}$. We have the equation

$$\bar{\mathtt{G}} = s_1 \mathtt{R} \left[\vec{t}_{-\delta}\right]_\times .$$

Notice that for $s_1 = 1$ we get Equation (2) and for $s_1 = -1$ we get Equation (1).

Thus, we get in total 4 solutions:

$$\left\{ \left(\underbrace{\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}}_{\mathtt{R}_+}, \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\vec{t}_{+\delta}} \right), \left(\underbrace{\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}}_{\mathtt{R}_-}, \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\vec{t}_{+\delta}} \right), \left(\underbrace{\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}}_{\mathtt{R}_+}, \underbrace{\begin{bmatrix}0\\0\\-1\end{bmatrix}}_{\vec{t}_{-\delta}} \right), \left(\underbrace{\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}}_{\mathtt{R}_-}, \underbrace{\begin{bmatrix}0\\0\\-1\end{bmatrix}}_{\vec{t}_{-\delta}} \right) \right\}.$$

We could look at the essential matrices $\mathtt{R}\left[\vec{t}_\delta\right]_\times$ these solutions generate:

$$\left\{ \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{\mathtt{R}_+\left[\vec{t}_{+\delta}\right]_\times}, \underbrace{\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&0\end{bmatrix}}_{\mathtt{R}_-\left[\vec{t}_{+\delta}\right]_\times}, \underbrace{\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&0\end{bmatrix}}_{\mathtt{R}_+\left[\vec{t}_{-\delta}\right]_\times}, \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{\mathtt{R}_-\left[\vec{t}_{-\delta}\right]_\times} \right\}$$

We see that all of them are multiples of $\mathtt{G}$. We can also check that $\mathtt{R}_+$ and $\mathtt{R}_-$ differ by the rotation about the baseline $\mathbf{b} = \langle \vec{t}_{+\delta} \rangle = \langle \vec{t}_{-\delta} \rangle$ by 180°:

$$\mathtt{R}_+ = \underbrace{\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}}_{\mathrm{Rot}_{\mathbf{b}}(180^\circ)} \mathtt{R}_-$$

The figure below shows the geometry of the computed 4 solutions [1, Figure 9.12]:

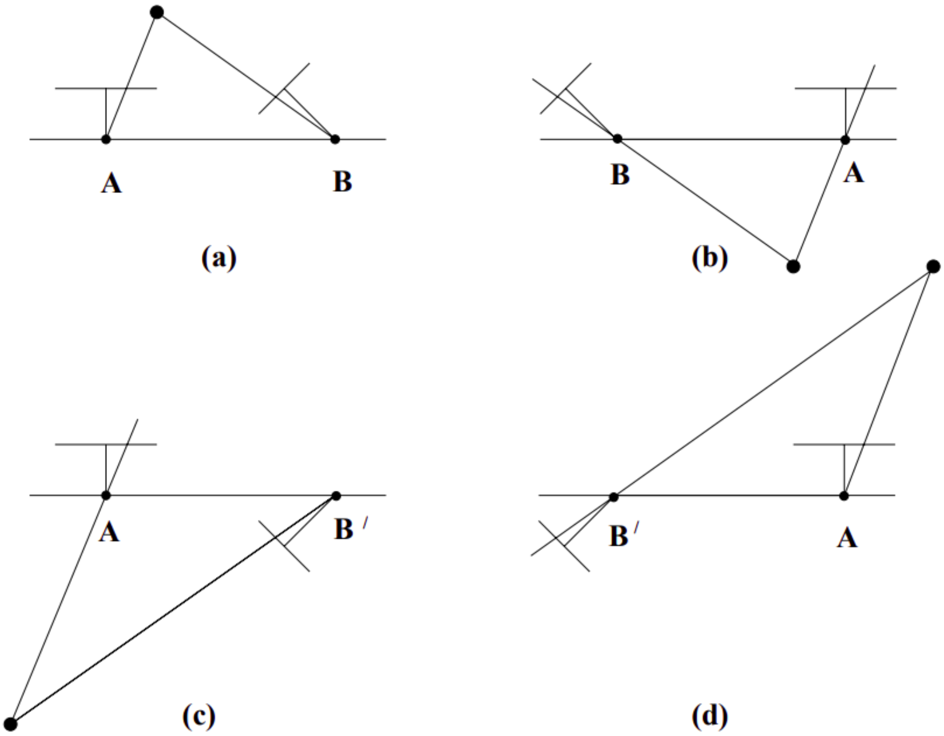

Fig. 9.12. **The four possible solutions for calibrated reconstruction from E.** *Between the left and right sides there is a baseline reversal. Between the top and bottom rows camera B rotates* 180° *about the baseline. Note, only in (a) is the reconstructed point in front of both cameras.*

□

References

[1] R. Hartley and A. Zisserman, *Multiple view geometry in computer vision*, Cambridge University Press, 2010, Second Edition.

[2] Tomas Pajdla, *Elements of geometry for computer vision*, `https://cw.fel.cvut.cz/wiki/_media/courses/gvg/pajdla-gvg-lecture-2021.pdf`.