SMU: Lecture 3

Monday, March 6, 2023

(Heavily inspired by the Stanford RL Course of Prof. Emma Brunskill, but all potential errors are mine.)

Plan for Today

- Recap of important concepts from lectures 1 and 2.
- Model-free control:
  - **Monte-Carlo Online Control**
  - **SARSA**
  - **Q-Learning**

Part 1: Where are we?

(Recap from the previous two lectures)

(Bellman equation for MDP)

State Value Function of MDP

Definition:

$$G_t^\pi = R(X_t, A_t) + \gamma \cdot R(X_{t+1}, A_{t+1}) + \gamma^2 \cdot R(X_{t+2}, A_{t+2}) + \ldots = \sum_{i=0}^{\infty} R(X_{t+i}, A_{t+i}) \cdot \gamma^i$$

$$V^\pi(s) = \mathbb{E}[G_t^\pi \,|\, X_t = s].$$

Computing it as a solution of a system of linear equations:

$$V^\pi(s) = \sum_{a \in A} \pi(a, s) \cdot \left[R(s, a) + \gamma \cdot \sum_{s' \in S} P(s' \,|\, s, a) \cdot V^\pi(s') \right]$$

Lecture 1

MDP Control Problem

How to find $\pi^*(s) = \arg\max_{\pi} V^{\pi}(s)$???

Lecture 1

MDP Control Problem

To be fully rigorous, we should write it like this, because there may be multiple optimal policies but only one optimal state-value function.

How to find $\pi^*(s) \in \arg\max_{\pi} V^{\pi}(s)$???

Lecture 1

State-Action Value Q

- **Definition:**

$$Q^{\pi}(s,a) = R(s,a) + \gamma \cdot \sum_{s' \in S} P(s' \mid s,a) \cdot V^{\pi}(s').$$

- **Intuition:**
  - The value of the return that we obtain if we first take the action a in the state s and then follow the policy π (including when we visit s again).
  - *Think of it as perturbing the policy π — we deviate from following the policy π only in the first step in s.*

Lecture 1

Policy Improvement Step

- **Given:** An MDP and a **policy π_i that we want to improve** (if possible).
- **DO:**
  - For all $s \in S$, compute $Q^{\pi_i}(s,a)$ as defined on the previous slide, i.e.
    $Q^{\pi_i}(s,a) = R(s,a) + \gamma \cdot \sum_{s' \in S} P(s' \mid s,a) \cdot V^{\pi_i}(s')$.
  - **Compute new policy for all $s \in S$:**

$$\pi_{i+1}(s) = \arg\max_{a \in S} Q^{\pi_i}(s,a)$$

Here, we use the fact that our policy is deterministic for simpler notation (treating policy as a function). Using our previous notation we could write:

$$\pi(a \mid s) = \begin{cases} 1 & \textit{if } a = \arg\max_{a \in A} Q^{\pi_i}(s,a) \\ 0 & \textit{otherwise} \end{cases}$$

Policy Iteration

$i = 0$

Initialize π_0 randomly.

DO

$V^{\pi_i} =$ Compute the state-value function, evaluating π_i.

$\pi_{i+1} =$ Policy improvement of π_i.

$i = i + 1$

WHILE $||\pi_i - \pi_{i-1}||_1 > 0$ /* if policy changed */

Policy iteration finds the globally optimal policy!

"Greedy Policy w.r.t. $Q^\pi(s,a)$"

Terminological note:

The policy satisfying

$$\pi'(s) = \arg\max_{a \in S} Q^\pi(s,a)$$

is called ***greedy policy w.r.t. the Q-function*** $Q^\pi(s,a)$**.**

(again, formally, we should be writing $\pi'(s) \in \arg\max_{a \in S} Q^\pi(s,a)$ *but we will just assume for simplicity thtat* $\arg\max$ *breaks ties in some consistent way and returns always only one state).*

Lecture 1

Value Iteration

Set $k = 1$

Initialize $V_0(s) = 0$ for all $s \in S$

DO:

$$V_k(s) = \max_{a \in A} \left[R(s,a) + \gamma \cdot \sum_{s' \in S} P(s' \mid s,a) \cdot V_{k-1}(s') \right]$$

Bellman backup B[V]

WHILE $\| V_k - V_{k-1} \|_\infty \geq \varepsilon$

- To extract an optimal policy, we can extract a deterministic (not necessarily unique) policy:

$$\pi(s) = \arg\max_{a \in A} \left[R(s,a) + \sum_{s' \in S} P(s' \mid s,a) \cdot V(s') \right].$$

Lecture 2

Problem: Model-Free Policy Evaluation

- Given a policy and an MDP with unknown parameters (or generally an environment with which we can interact), **estimate the value function.**

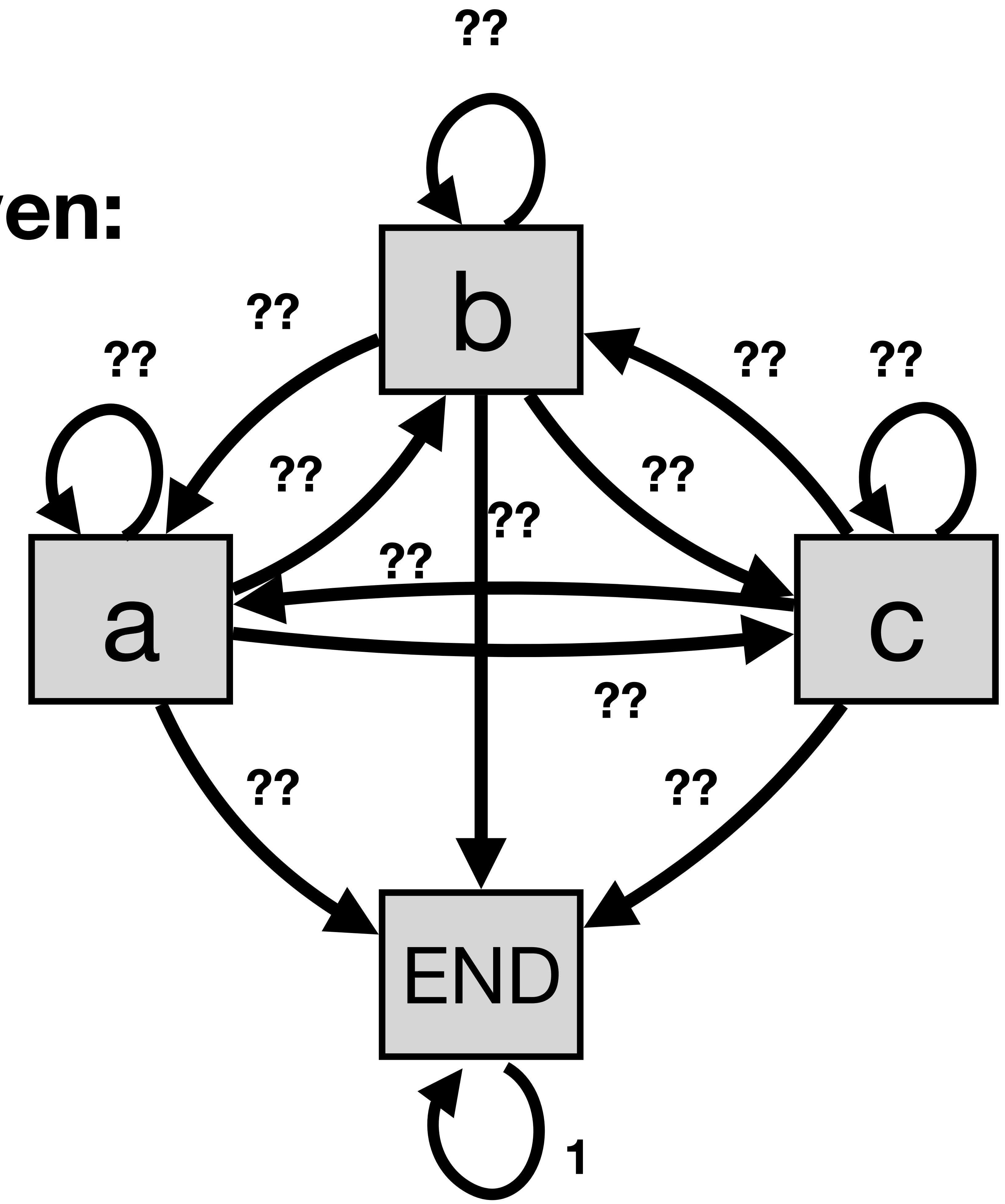
Lecture 2

Example

Agent:

Rewards??

States are given:



Actions are given:

$$A = \{l, r\}$$

Policy is given, e.g.:

$$\pi(l \mid a) = 0.2, \pi(r \mid a) = 0.8,$$
$$\pi(l \mid b) = 0.3, \pi(r \mid b) = 0.7,$$

...

First/Every-Visit Monte-Carlo Evaluation

Initialize: $G(s) = 0$, $N(s) = 0$, $V^\pi(s) = \textit{undefined}$ **for all** $s \in S$.

For $i = 1,\ldots,N$**:**

Sample episode $e_i := s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$.

For each time step $1 \leq t \leq T_i$:

If t is the first occurrence of state s in the episode e_i

$g_{i,t} := r_{i,t} + \gamma \cdot r_{i,t+1} + \gamma^2 \cdot r_{i,t+2} + \ldots + \gamma^{T_i - t} \cdot r_{i,T_i}$

$N(s) := N(s) + 1$ /* *Increment total visits counter* */

$G(s) := G(s) + g_{i,t}$ /* *Increment total return counter* */

$V^\pi(s) := G(s)/N(s)$ /* *Update current estimate* */

Lecture 2

Temporal Difference Learning

- **TD learning** combines Monte-Carlo estimation and dynamic programming ideas.
- **TD learning** can be used both in episodic and infinite-horizon non-episodic settings,
- **TD learning** updates estimates of V^{π} continually, after every consecutive tuple *state-action-reward-state* (therefore we do not need to wait till the end of an episode).

….

TD-Learning: Pseudocode

Initialize: $V^\pi(s) = 0$ for all $s \in S$

Loop:

Sample tuple (s_t, a_t, r_t, s_{t+1}).

Update $V^\pi(s_t) := V^\pi(s_t) + \alpha \cdot (\underbrace{r_{i,t} + \gamma \cdot V^\pi(s_{t+1})}_{\text{TD target}} - V^\pi(s_t))$

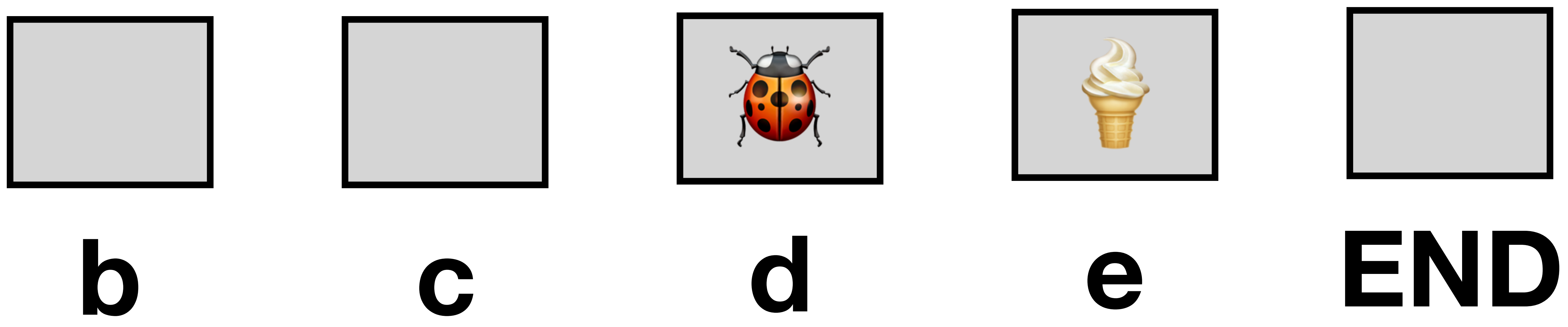
Part 2: Model-Free Control
(Problem Statement)

Model-Free Control

- Given an MDP with unknown parameters (or generally an environment with which we can interact), **find an optimal policy π.**

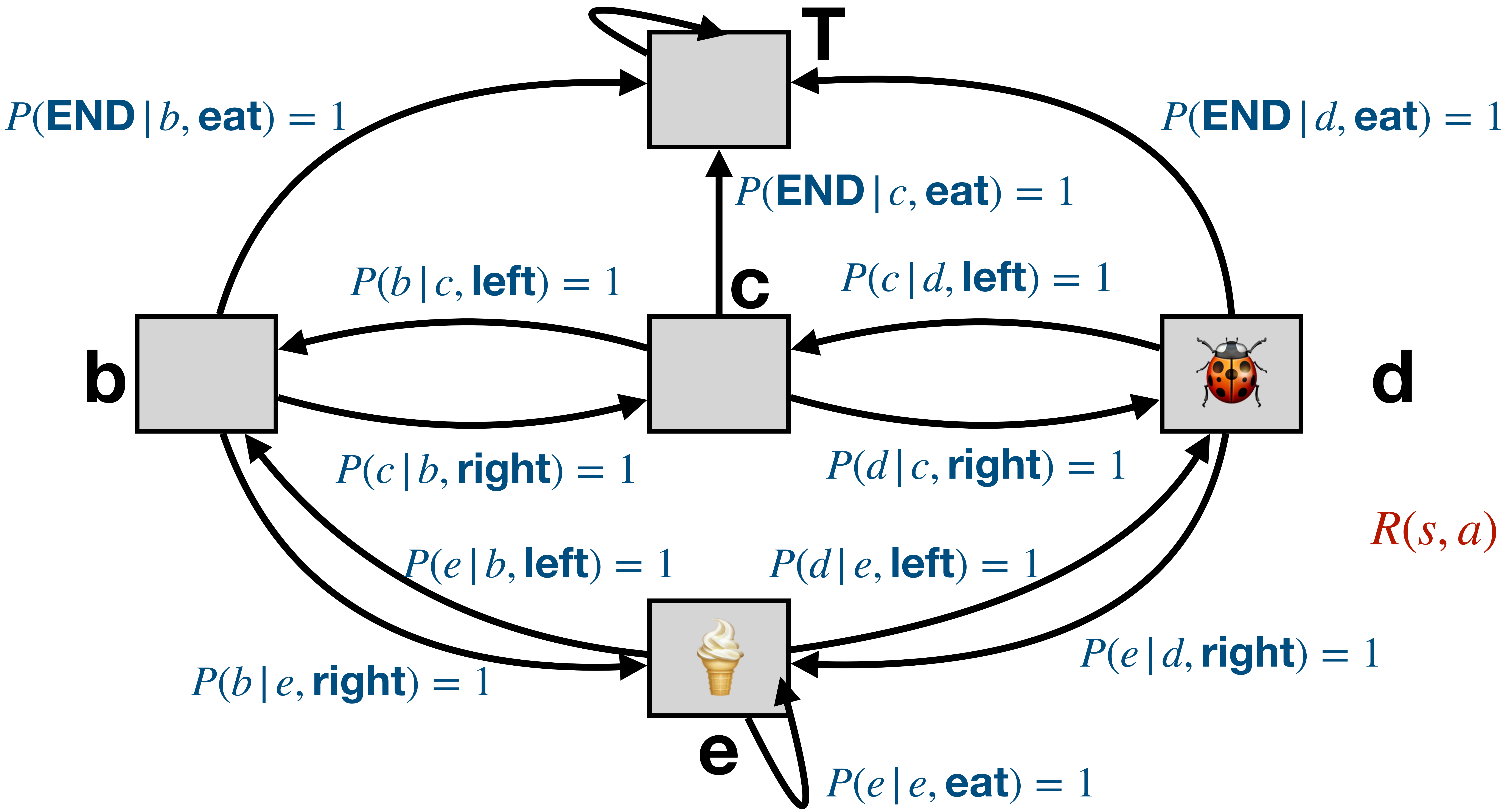
Running Example

- **Example we will use:**
  - Agent (ladybug)
  - State space: $S = \{b, c, d, e, \text{END}\}$, END is the terminal state.
  - Action space: $A = \{\text{left}, \text{right}, \text{eat}\}$.
  - We **do not know** $P(s' \mid s, a)$, $R(s, a)$ and $\pi(a \mid s)$.
  - We want to learn some optimal policy!



Running Example

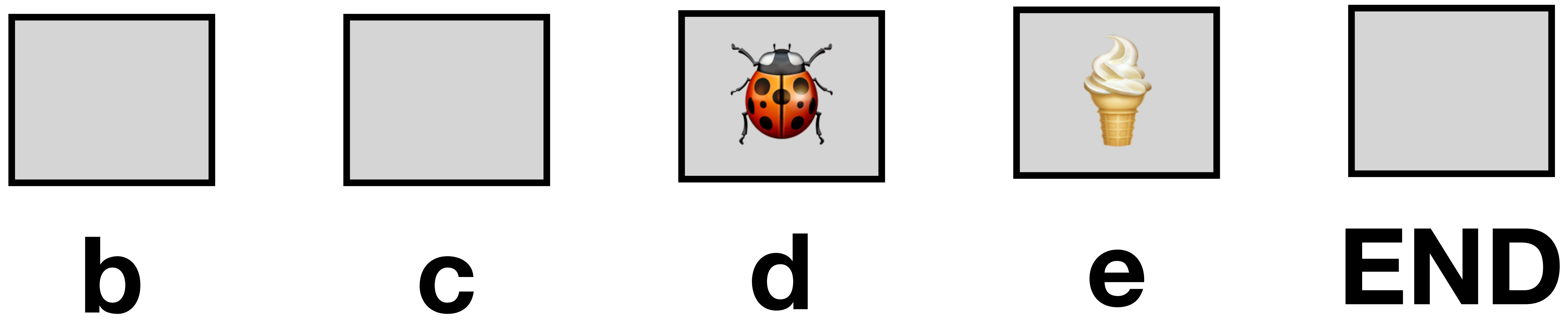
- Here, is what the system will behave like - this is just for you to have some intuition, the RL algorithm will not have access to this information.



$$R(s,a) = \begin{cases} 1 & s \in \{b, c, d\} \\ 10 & s = e \textbf{ and } a = \textbf{eat} \\ 0 & \textbf{otherwise} \end{cases}$$

Part 3: Model-Free Policy Iteration

An Idea



- What if we wanted to use policy iteration to find the optimal policy?
- What would we need?
- **Answer:** We would need to be able to compute the state-action value function $Q^{\pi}(s, a)$ for any policy π. But that's not possible because we do not know the parameters of the MDP…
- **Idea:** Could we estimate $Q^{\pi}(s, a)$ in a similar way as we were estimating $V^{\pi}(s)$ last week? And then use policy improvement on that estimated $Q^{\pi}(s, a)$?

MC Estimation of $Q^{\pi}(s, a)$

Last time we talked about MC Estimation of the value function.

We can use the same idea for the estimation of the state-action value function $Q^{\pi}(s, a)$…

…then use that estimated $Q^{\pi}(s, a)$ as in policy iteration…

MC Estimation of $Q^{\pi}(s, a)$

Last time we talked about MC Estimation of the value function.

We can use the same idea for the estimation of the state-action value function $Q^{\pi}(s, a)$…

…then use that estimated $Q^{\pi}(s, a)$ as in policy iteration…

…and see how it fails if done naively.

A Naive Idea

- **THIS WILL NOT WORK (YET):**

Initialize: $G(s, a) = 0$, $N(s, a) = 0$ **for all** $s \in S$, $\pi_1 = \pi$ (*the given policy*).

For $i = 1, \ldots, N$:

Sample episode $e_i := s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$ using π_i.

For each time step $1 \leq t \leq T_i$:

(**If** t is the first occurrence of state s in the episode e_i - Use this if you want first-visit MC)

s_t is the state visited at time t in the episode e_i

a_t is the action taken at time t in the episode e_i

$g_{i,t} := r_{i,t} + \gamma \cdot r_{i,t+1} + \gamma^2 \cdot r_{i,t+2} + \ldots + \gamma^{T_i - t} \cdot r_{i,T_i}$

$N(s) := N(s) + 1$ /* *Increment total visits counter* */

$G(s_t, a_t) := G(s_t, a_t) + g_{i,1}$ /* *Increment total return counter* */

$Q(s_t, a_t) := G(s_t, a_t) / N(s_t, a_t)$ /* *Update current estimate* */

Set $\pi_{i+1} =$ *greedy policy w.r.t.* Q, *i.e.*, $\pi(s) = \arg\max_{a \in A} Q(s, a)$ /* *breaking ties consistently* */.

Let's see why it will not work!

$S = \{b, c, d, e, \text{END}\}, A = \{\text{left, right, eat}\}$

$\pi(b) = \pi(c) = \pi(e) = \text{left}, \pi(d) = \text{eat}$

$e_1 = c, \text{left}, 1, b, \text{left}, 1, e, \text{left}, 1, d, \text{eat}, 0, \text{END}$

How can we ever estimate, e.g., $Q^\pi(b, \text{right})$??

The problem is we may never update the estimate for $Q^\pi(b, \text{right})$ because the action taken in the state b is always left.

- **A simple idea** (that will not work yet… and will illustrate why we need to think about exploration)**:**
  - **THIS WILL NOT WORK (YET):**

Initialize: $G(s, a) = 0, N(s, a) = 0$ **for all** $s \in S$.

For $i = 1, \dots, N$**:**

Sample episode $e_i := s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \dots, s_{i,T_i}$ using π.

For each time step $1 \le t \le T_i$:

(**If** t is the first occurrence of state s in the episode e_i
- Use this if you want first-visit MC)

s_t is the state visited at time t in the episode e_i

a_t is the action taken at time t in the episode e_i

$g_{i,t} := r_{i,t} + \gamma \cdot r_{i,t+1} + \gamma^2 \cdot r_{i,t+2} + \dots + \gamma^{T_i - t} \cdot r_{i,T_i}$

$N(s) := N(s) + 1$ /* *Increment total visits counter* */

$G(s_t, a_t) := G(s_t, a_t) + g_{i,1}$ /* *Increment total return counter* */

$Q^\pi(s_t, a_t) := G(s_t, a_t)/N(s_t, a_t)$ /* *Update current estimate* */

ε-Greedy Policy

- Given a Q-function $Q(s, a)$, we define the ε-greedy policy w.r.t. Q as

We assume ties are decided consistently

$$\pi(a \mid s) = \begin{cases} 1 - \varepsilon + \frac{\varepsilon}{|A|} & \text{when } a = \arg\max_{a \in A} Q(s, a) \\ \frac{\varepsilon}{|A|} & \text{when } a \neq \arg\max_{a \in A} Q(s, a) \end{cases}$$

MC On Policy Iteration

Initialize: $G(s, a) = 0$, $N(s, a) = 0$, $Q(s, a) = 0$ **for all** $s \in S, a \in A$.

Initialize: $\varepsilon = 1$, $k = 1$

For $i = 1, \ldots, N$:

Sample episode $e_i := s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$ **given π_k.**

For each time step $1 \leq t \leq T_i$:

(**If** t is the first occurrence of state s in the episode e_i - Use this if you want first-visit MC)

s_t is the state visited at time t in the episode e_i

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$G(s_t, a_t) := G(s_t, a_t) + g_{i,1}$ /* *Increment total return counter* */

$Q(s_t, a_t) := G(s_t, a_t)/N(s_t, a_t)$ /* *Update current estimate* *

EndFor

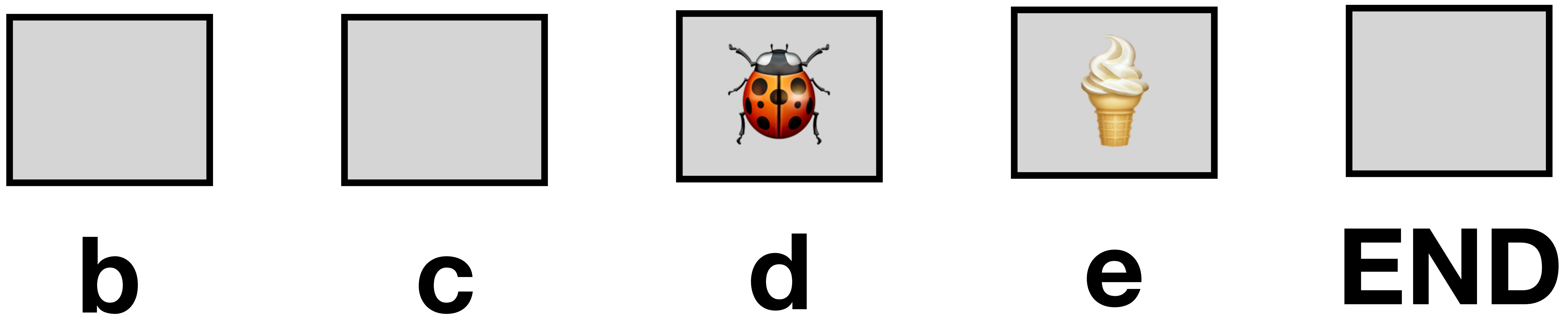
$k = k + 1$, $\varepsilon = 1/k$

$\pi_k = \varepsilon-$**greedy policy w.r.t.** Q

Running Example (Initialization)

Let's run MC On-Policy Iteration on our running example ($\gamma = 0.5$):

$k = 1, \varepsilon = 1$



G(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

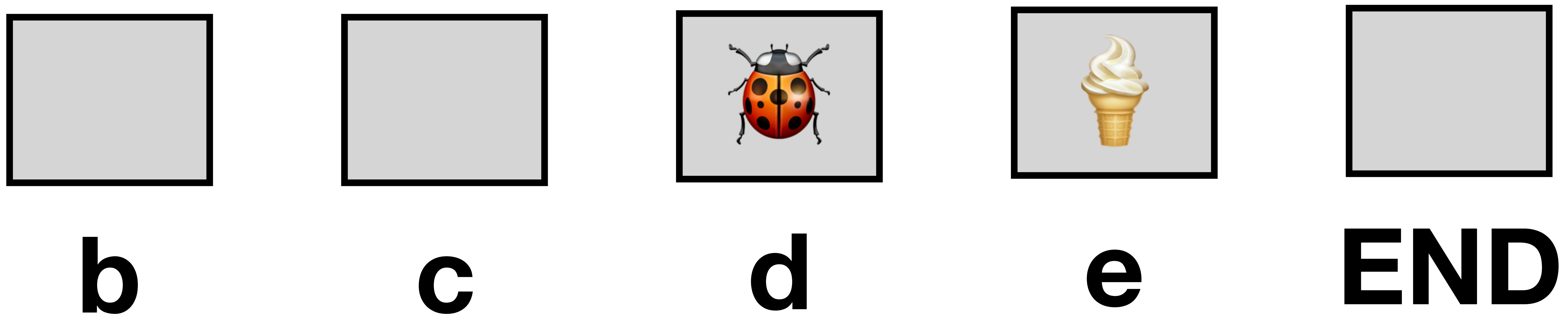
N(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example (Episode 1)

Let's run MC On-Policy Iteration on our running example ($\gamma = 0.5$):

$k = 1, \varepsilon = 1$



$e_1 = d$

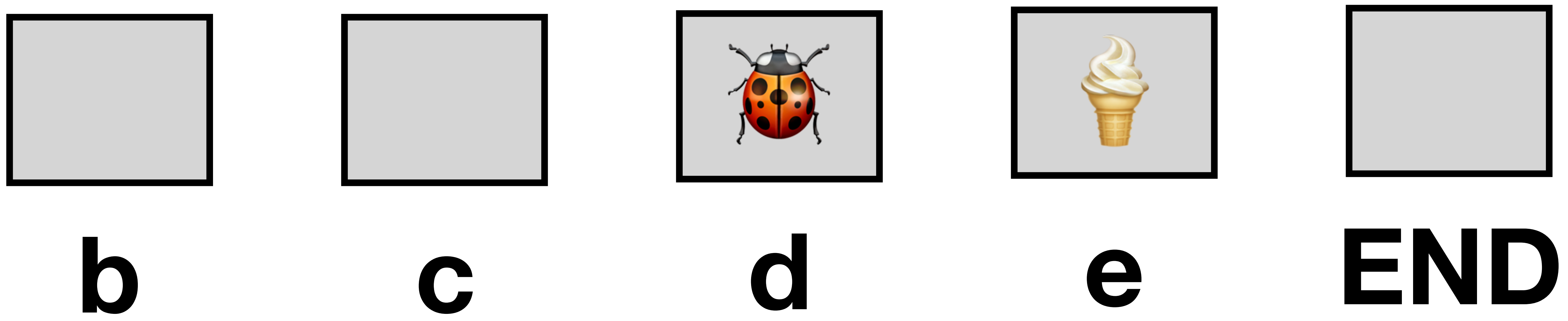
Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

$$\pi_1(a \mid d) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Running Example (Episode 1)

Let's run MC On-Policy Iteration on our running example ($\gamma = 0.5$):

$k = 1,\ \varepsilon = 1$



$e_1 = d,$ right

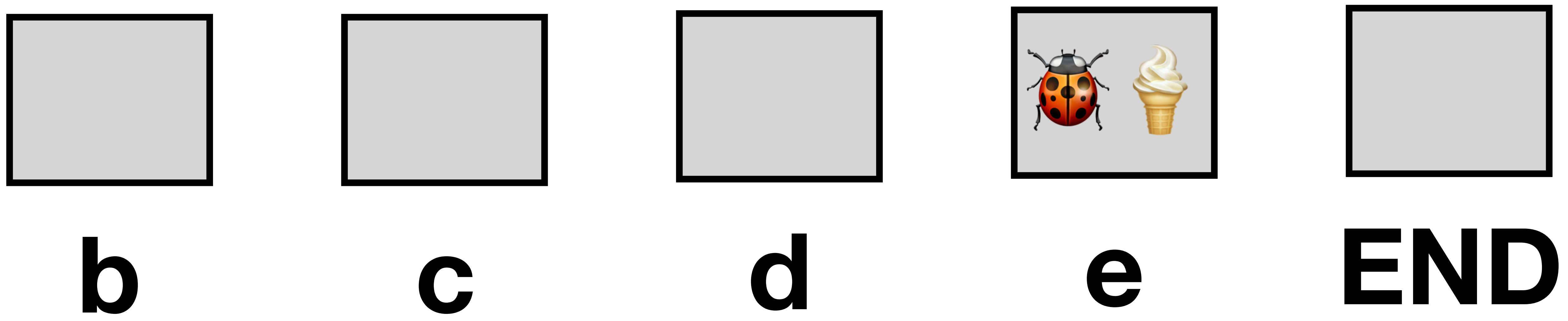
$$\pi_1(a \mid d) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example (Episode 1)

Let's run MC On-Policy Iteration on our running example ($\gamma = 0.5$):

$k = 1,\ \varepsilon = 1$



$e_1 = d,$ right, $1,$e

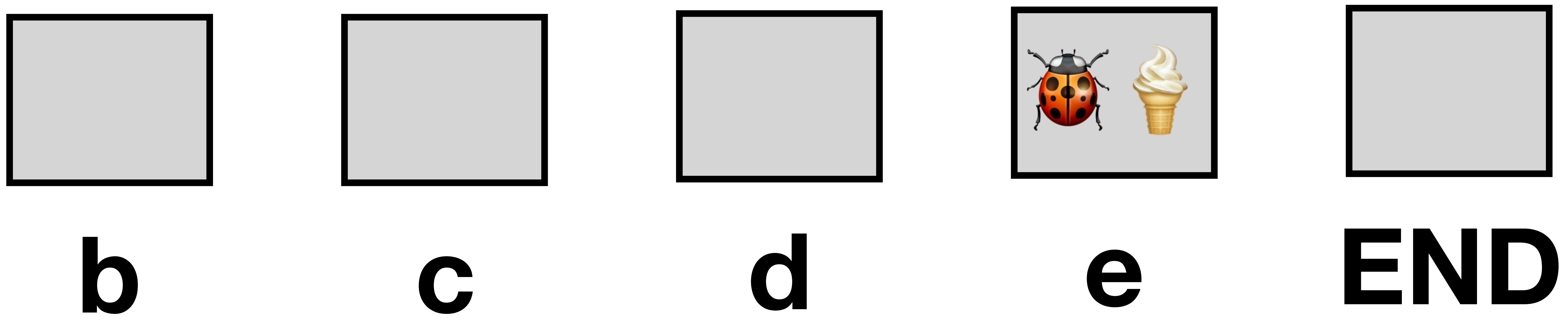
Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

$$\pi_1(a \mid e) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Running Example (Episode 1)

Let's run MC On-Policy Iteration on our running example ($\gamma = 0.5$):

$k = 1,\ \varepsilon = 1$



$e_1 = d,$ right$, 1,$e, right

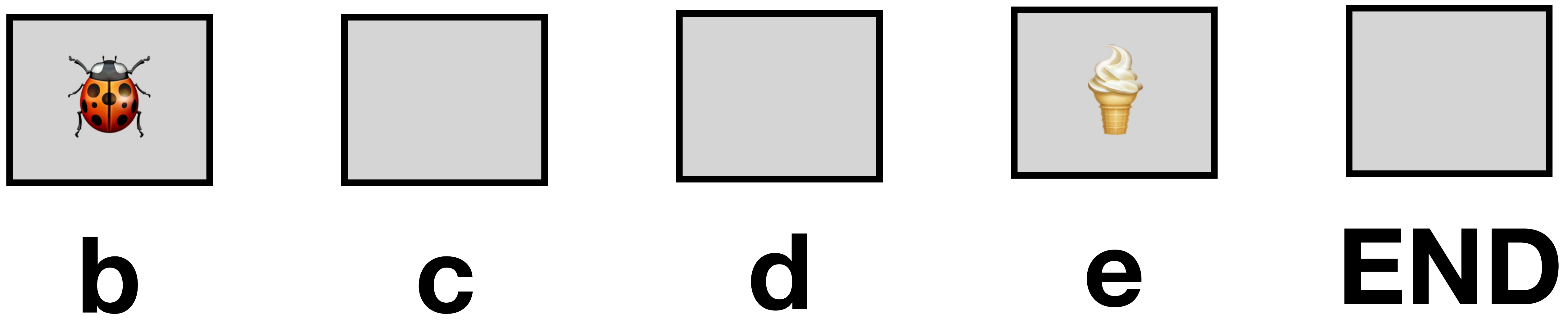
$$\pi_1(a \mid e) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example (Episode 1)

Let's run MC On-Policy Iteration on our running example ($\gamma = 0.5$):

$k = 1,\ \varepsilon = 1$



$e_1 = d$, right, 1, e, right, 1, b

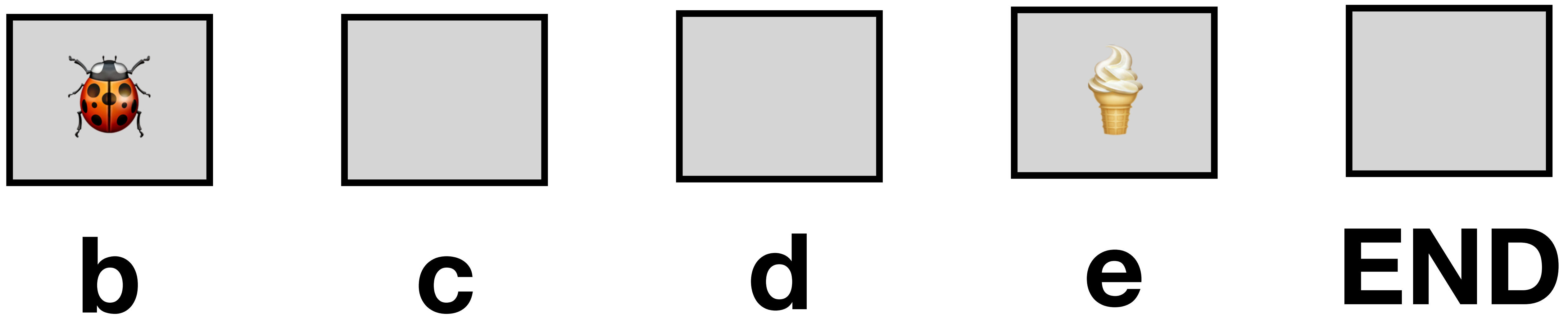
$$\pi_1(a \mid b) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example (Episode 1)

Let's run MC On-Policy Iteration on our running example ($\gamma = 0.5$):

$k = 1,\ \varepsilon = 1$



$e_1 = d,$ right$,1,$e, right$,1,b,$eat

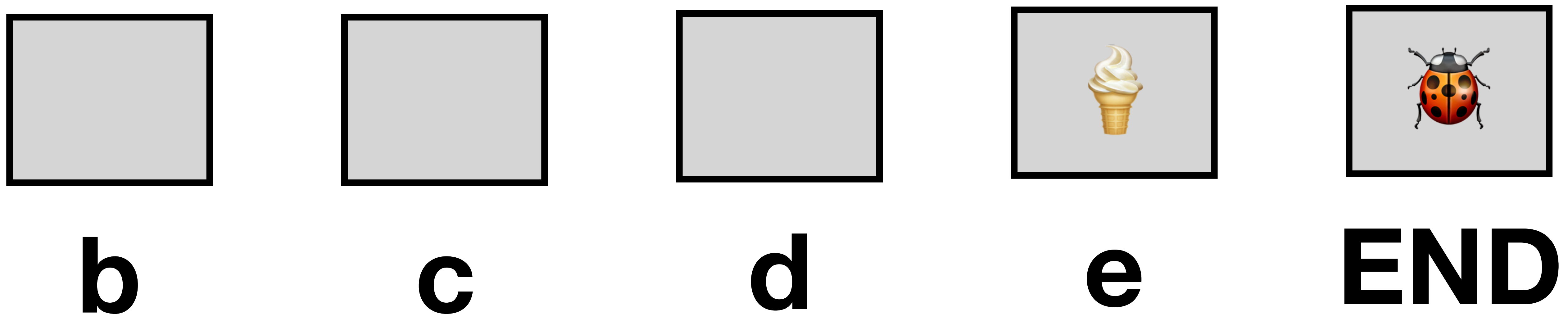
$$\pi_1(a \mid b) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example (Episode 1)

Let's run MC On-Policy Iteration on our running example ($\gamma = 0.5$):

$k = 1, \varepsilon = 1$



$e_1 = d,$ right$, 1,$ e$,$ right$, 1, b,$ eat$, 0,$ END

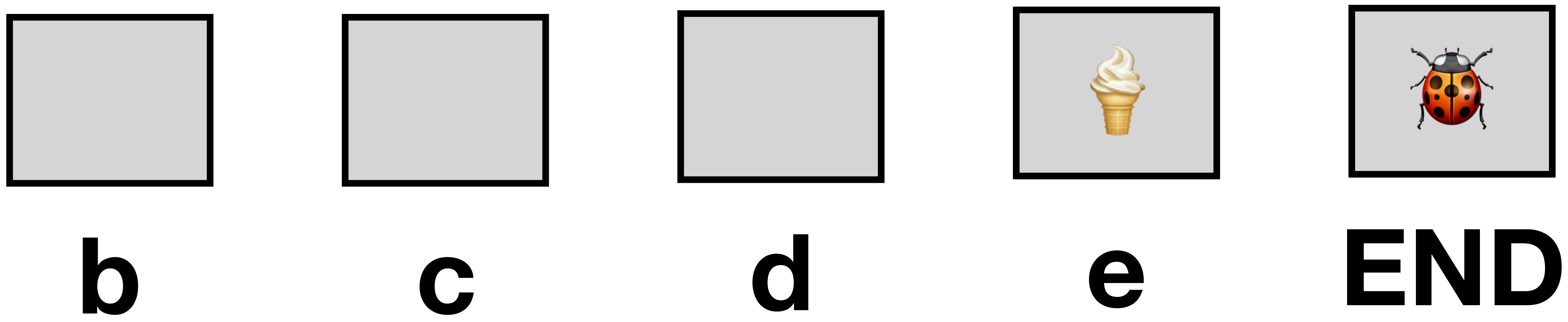
Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example (Episode 1)

Now we use First-Visit MC to update G, N and Q.

$k = 1,\ \varepsilon = 1$

$e_1 = d, \text{right}, 1, \text{e}, \text{right}, 1, b, \text{eat}, T$



G(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	1.5	0
e	0	1	0

N(s,a)	*left*	*right*	*eat*
b	0	0	1
c	0	0	0
d	0	1	0
e	0	1	0

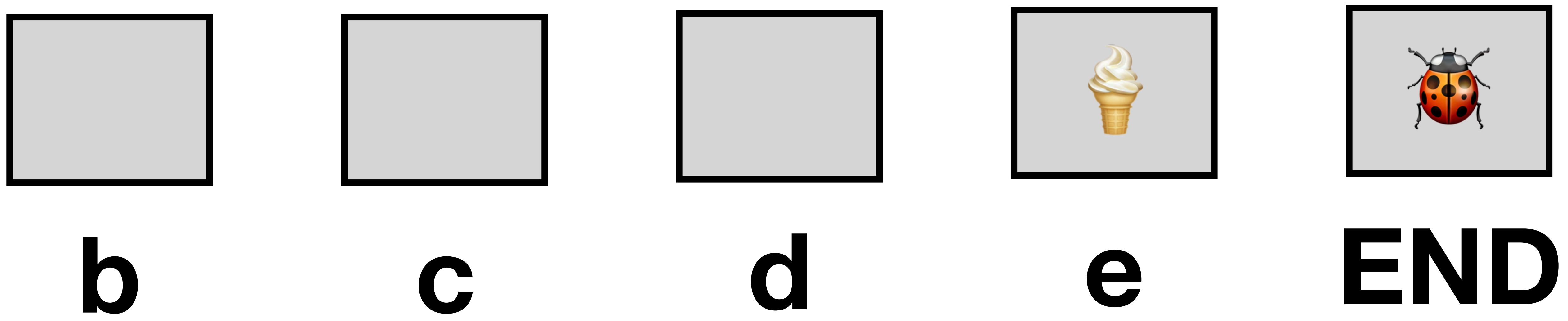
Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	1.5	0
e	0	1	0

Running Example (Episode 1)

Now we use First-Visit MC to update G, N and Q.

$k = 1,\ \varepsilon = 1$

$e_1 = d,\ \text{right},\ 1, \text{e},\ \text{right}, 1, b, \text{eat},\ T$



G(s,a)	*left*	*right*	*eat*
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c	0	0	0
d	0	1.5	0
e	0	1	0

N(s,a)	*left*	*right*	*eat*
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c	0	0	0
d	0	1	0
e	0	1	0

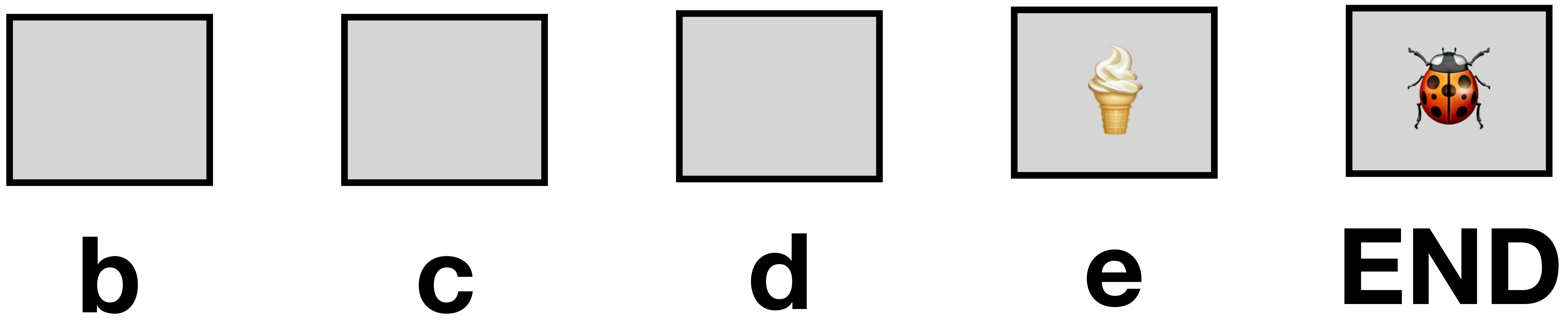
Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	1.5	0
e	0	1	0

Running Example (Episode 1)

Now we use First-Visit MC to update G, N and Q.

$k = 1,\ \varepsilon = 1$

$e_1 = d,$ right, 1, e, right, $1, b,$ eat, T



G(s,a)	*left*	*right*	*eat*
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d	0	1.5	0
e	0	1	0

N(s,a)	*left*	*right*	*eat*
b	0	0	1
c	0	0	0
d	0	1	0
e	0	1	0

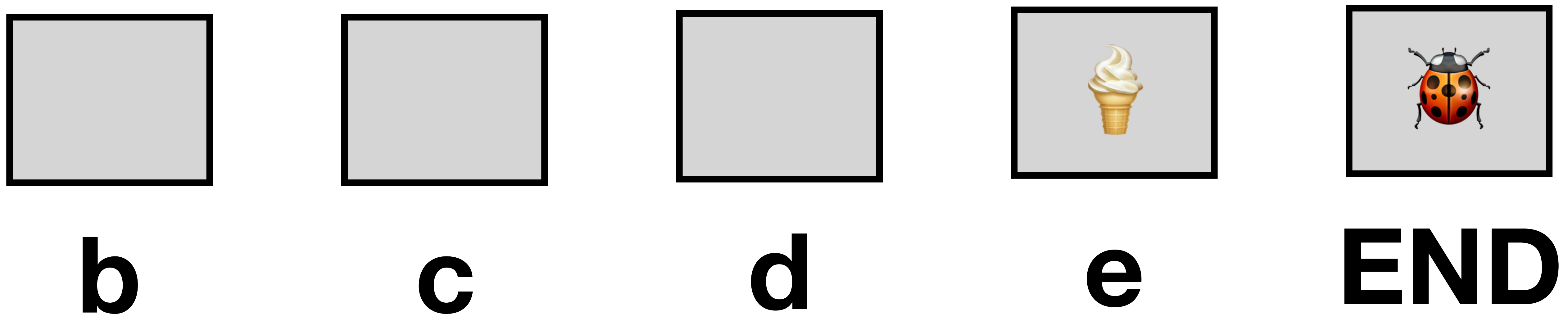
Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	1.5	0
e	0	1	0

Running Example (Episode 1)

Now we use First-Visit MC to update G, N and Q.

$k = 1,\ \varepsilon = 1$

$e_1 = d,$ right$,1,$e, right$,1,b,$eat$,T$



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c	0	0	0
d	0	1.5	0
e	0	1	0

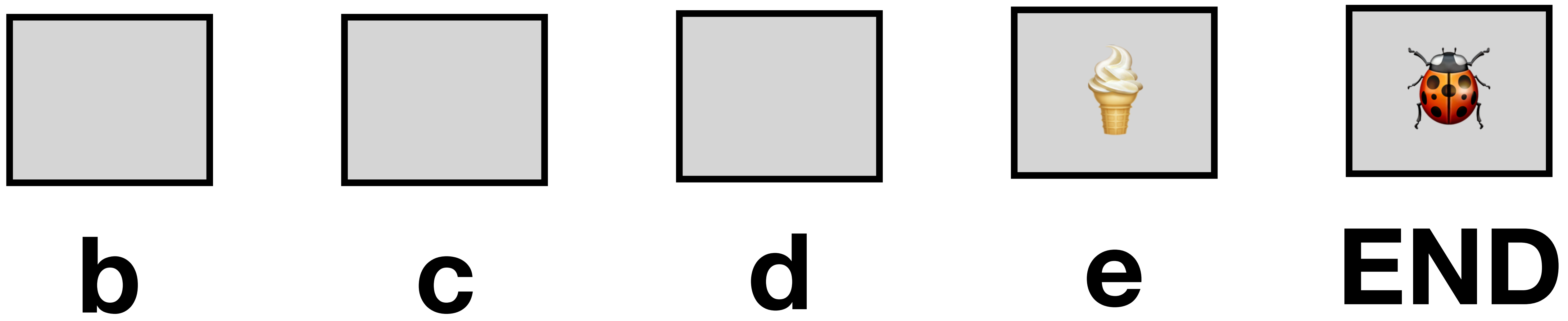
N(s,a)	*left*	*right*	*eat*
b	0	0	1
c	0	0	0
d	0	1	0
e	0	1	0

Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	1.5	0
e	0	1	0

Running Example (Episode 1)

Now we update the policy π. First, we get the greedy policy w.r.t. $Q(s, a)$

$k = 1,\ \varepsilon = 1$



Let us suppose that if there is tie in $\arg\max_{a \in A}$ then the preference is eat $\prec$ right $\prec$ left (i.e. we prefer left over right and right over eat)

$$\pi_{\text{greedy}}(d) = \pi_{\text{greedy}}(e) = \text{right},$$

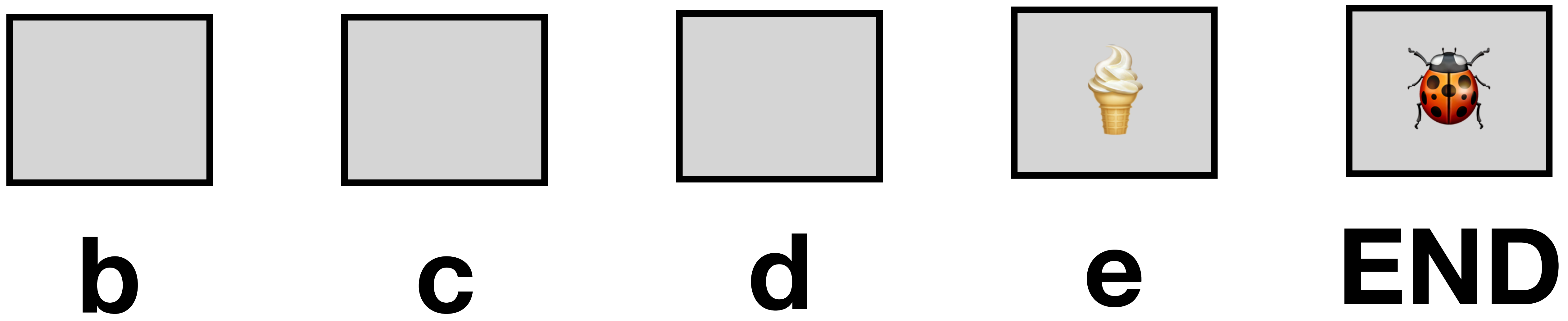
$$\pi_{\text{greedy}}(b) = \pi_{\text{greedy}}(c) = \text{left}.$$

Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	1.5	0
e	0	1	0

Running Example (Episode 1)

Now we update the policy π. First, we get the greedy policy w.r.t. $Q(s, a)$

Now, we update $k = 2; \varepsilon = 0.5$

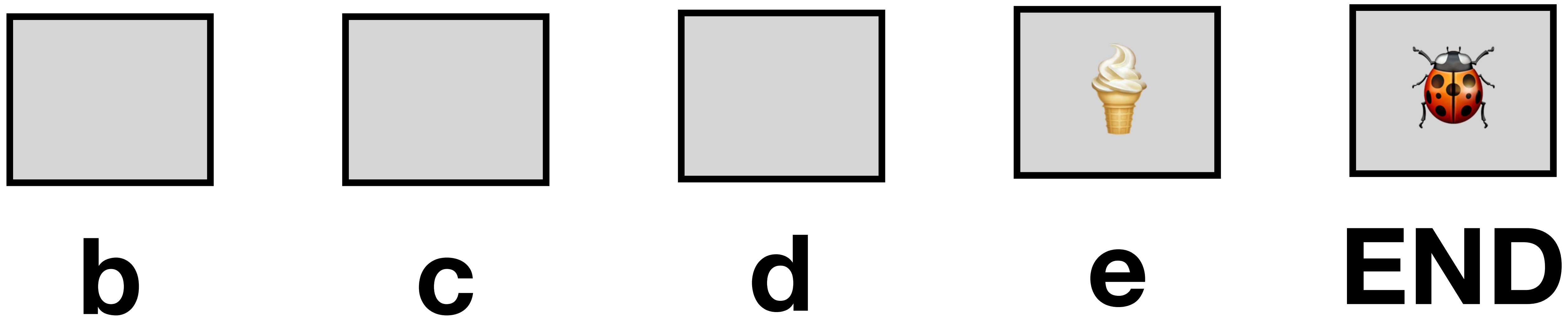


The new policy π will be the ε-greedy policy:

$$\pi(a \mid s) = \begin{cases} 1 - \varepsilon + \frac{\varepsilon}{|A|} & \text{when } a = \pi_{\text{greedy}}(s) \\ \frac{\varepsilon}{|A|} & \text{when } a \neq \pi_{\text{greedy}}(s) \end{cases}$$

We then run the next iteration with this new policy π.

Running Example (Episode 1)



As k increases, the algorithm will converge to the optimal policy:

$$\pi(b) = \textbf{left}, \pi(c) = \textbf{left}, \pi(d) = \textbf{right}, \pi(e) = \textbf{eat}$$

GLIE

- We say that an algorithm has the GLIE property (= "greedy in the limit of infinite exploration"), if it satisfies the following two conditions):

- **Definition** (GLIE conditions**):**

  1. If a state $s \in S$ is visited infinitely often, then each action in that state is chosen infinitely often (with probability 1)

  2. In the limit (as t → ∞), the learning policy is greedy with respect to the learned Q-function (with probability 1). By *greedy* we mean (ignoring the possibility of ties in the $\arg\max$ for simplicity) that

$$\pi_{k+1}(a \mid s) = \begin{cases} 1 & \text{for } a = \arg\max_{a \in A} Q_k(s, a), \\ 0 & \text{otherwise.} \end{cases}$$

MC Policy Iteration with $\varepsilon_i = 1/i$ is GLIE

- For a proof, see, e.g. *Singh, S., Jaakkola, T., Littman, M. L., & Szepesvári, C. (2000). Convergence results for single-step on-policy reinforcement-learning algorithms. Machine learning, 38(3), 287-308.*

- The formal proof is a bit tricky…

- **Note:** *There are other seauences of* ε_i *which guarantee GLIE as well.*

A Theorem (Why GLIE Matters)

- **Theorem**: GLIE Monte-Carlo Control converges to the optimal state-action value function, i.e. $Q_k(s, a) \to Q^*(s, a)$ as $k \to \infty$.

Part 4: SARSA and Q-Learning

General Form of TD-Based Methods

- **Basic idea:**
  - Replace Monte Carlo Policy Evaluation by a temporal-difference method.
  - Still use ε-greedy policies to guarantee that exploration will take place.

Bellman Equations for Q-Function

(Something we skipped when we talked about Q-functions for MDPs but something that will be useful now.)

We have:

$$V^\pi(s) = \sum_{a \in A} \pi(a \mid s) \cdot Q^\pi(s, a)$$

$$Q^\pi(s, a) = R(s, a) + \gamma \cdot \sum_{s' \in S} P(s' \mid s, a) \cdot V^\pi(s')$$

Combining the above:

$$Q^\pi(s, a) = R(s, a) + \gamma \cdot \sum_{s' \in S} P(s' \mid s, a) \cdot \sum_{a' \in A} \pi(a' \mid s') \cdot Q^\pi(s', a')$$

TD-Target

Bellman for Q-function:

$$Q^\pi(s_t, a_t) = R(s_t, a_t) + \gamma \cdot \boxed{\sum_{s_{t+1} \in S} P(s_{t+1} \mid s_t, a_t) \cdot \sum_{a_{t+1} \in A} \pi(a_{t+1} \mid s_{t+1}) \cdot Q^\pi(s_{t+1}, a_{t+1})}$$

$$\mathbb{E}[Q^\pi(X_{t+1}, A_{t+1}) \mid X_t = s_t, A_t = a_t]$$

Temporal difference update (SARSA)…

$$Q(s_t, a_t) := Q(s_t, a_t) + \alpha \left(\boxed{r_t + \gamma Q(s_{t+1}, a_{t+1})} - Q(s_t, a_t) \right)$$

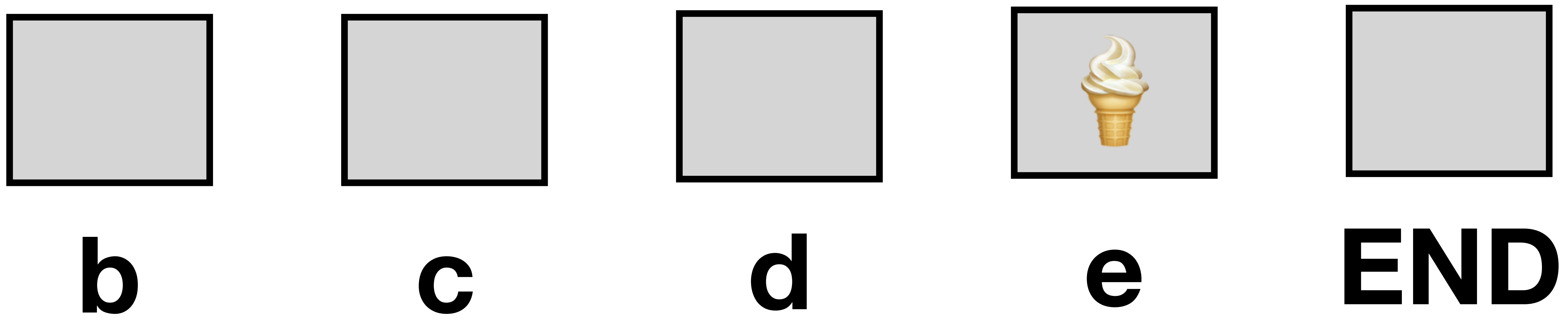
SARSA

1. **Initialize:** set π to be some ε-greedy policy, set $t = 1$, initialize $Q(s, a)$.
2. **Sample** a_1 using the distribution given by π in the state s_1 *(for sampling, we will use the notation* $a_1 \sim \pi(s_1)$*)*.
3. **While** s_t is not a terminal state:
   1. **Take** action a_t and observe r_t, s_{t+1}.
   2. **Sample** $a_{t+1} \sim \pi(s_{t+1})$ and store it for the next iteration.
   3. $Q(s_t, a_t) := Q(s_t, a_t) + \alpha \left(r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t) \right)$
   4. $\pi := \varepsilon$-greedy(Q)
   5. Set $t := t + 1$. Update ε, α /* *see next slides* */

Running Example (Initialization)

Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.
$t = 1, \varepsilon = 1, \alpha = 0.1$

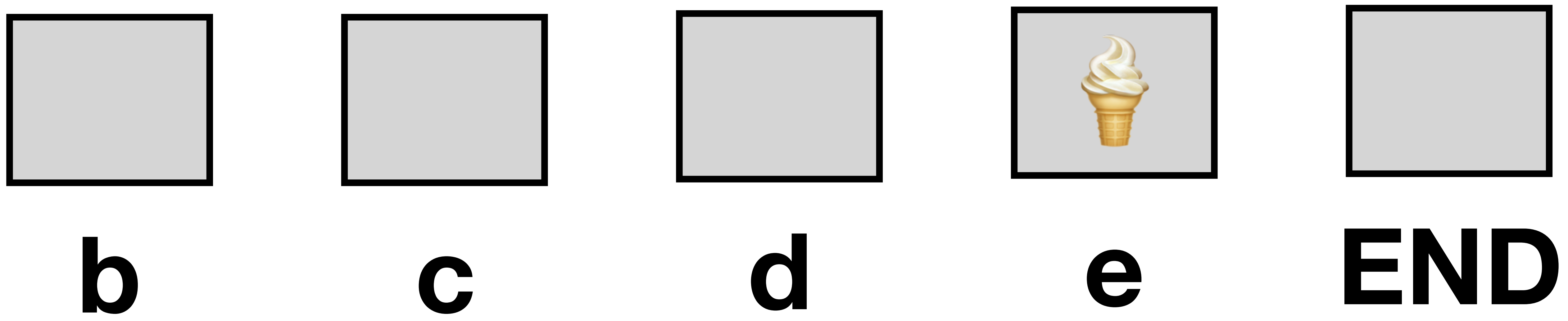


Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example (Initialization)

Let’s run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.
$t = 1,\ \varepsilon = 1,\ \alpha = 0.1$

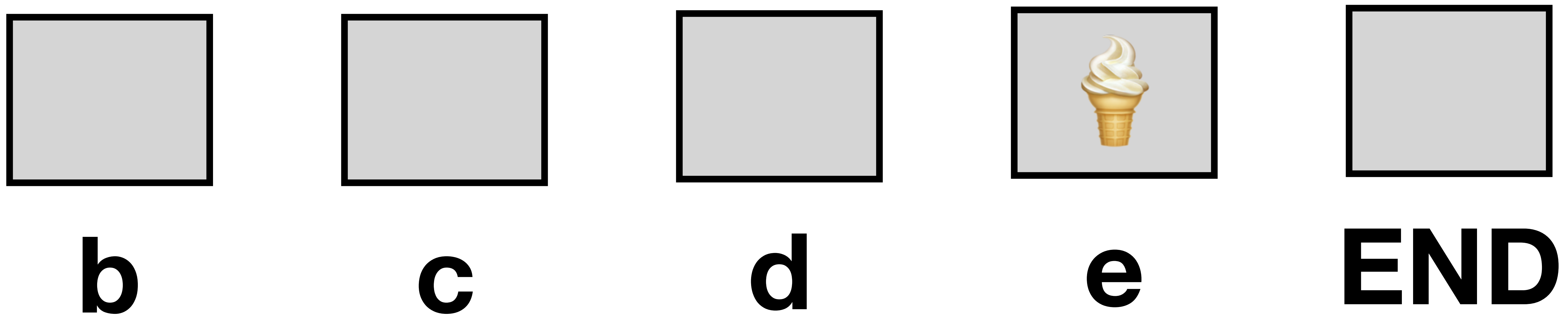


Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example (Initialization)

Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.
$t = 1, \varepsilon = 1, \alpha = 0.1$



Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

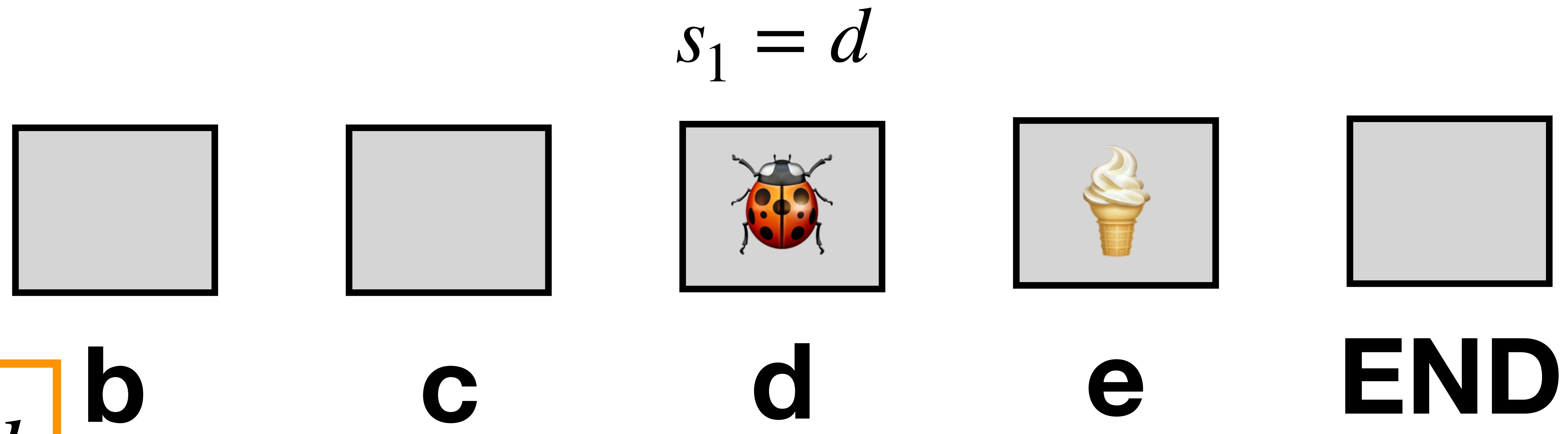
Running Example (Initialization)

Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.

$t = 1$, $\varepsilon = 1$, $\alpha = 0.1$

World samples the state $s_1 = d$



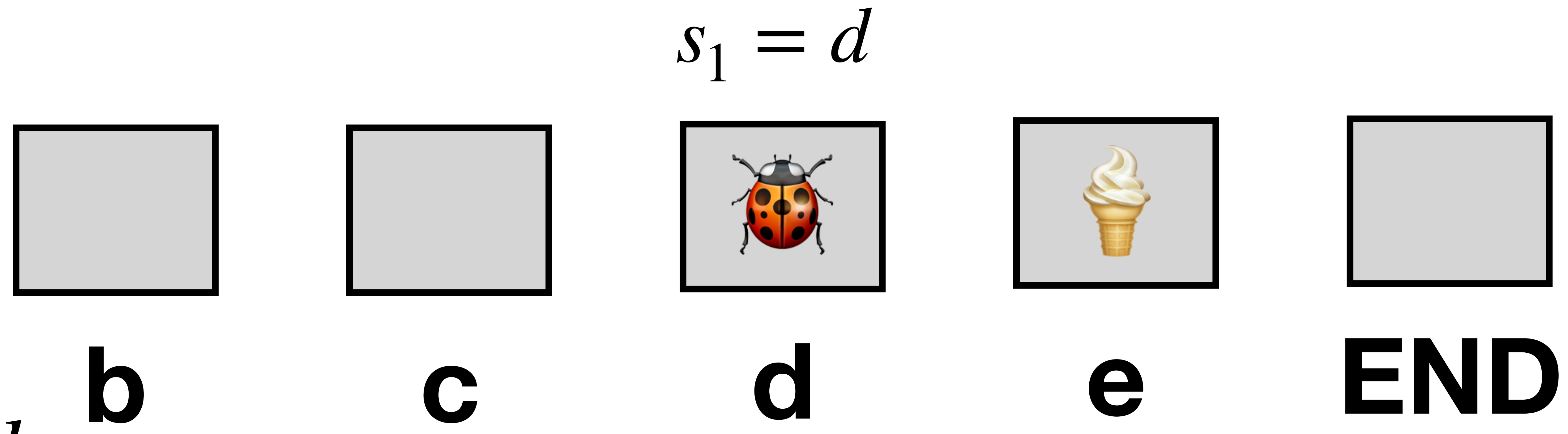
Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example (Initialization)

Let’s run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.

$t = 1,\ \varepsilon = 1,\ \alpha = 0.1$



World samples the state $s_1 = d$

We sample a_1 (we do not take it yet)

$$a_1 \sim \pi_1(a \mid d) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example ($t = 1$)

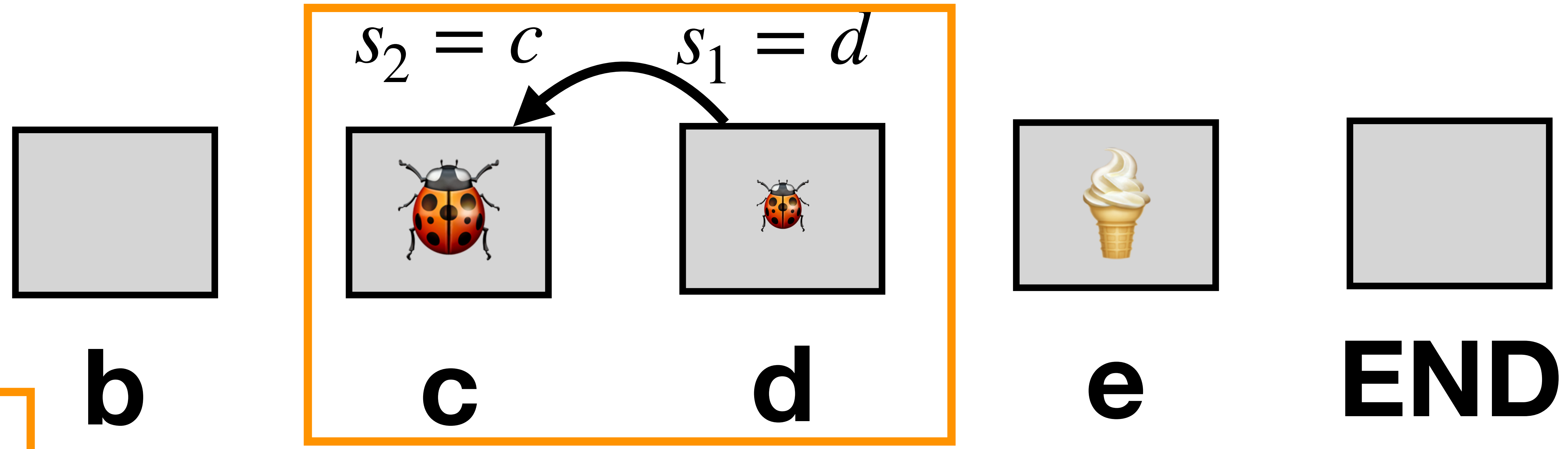
Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.

$t = 1$, $\varepsilon = 1$, $\alpha = 0.1$

We take the action a_1 = left

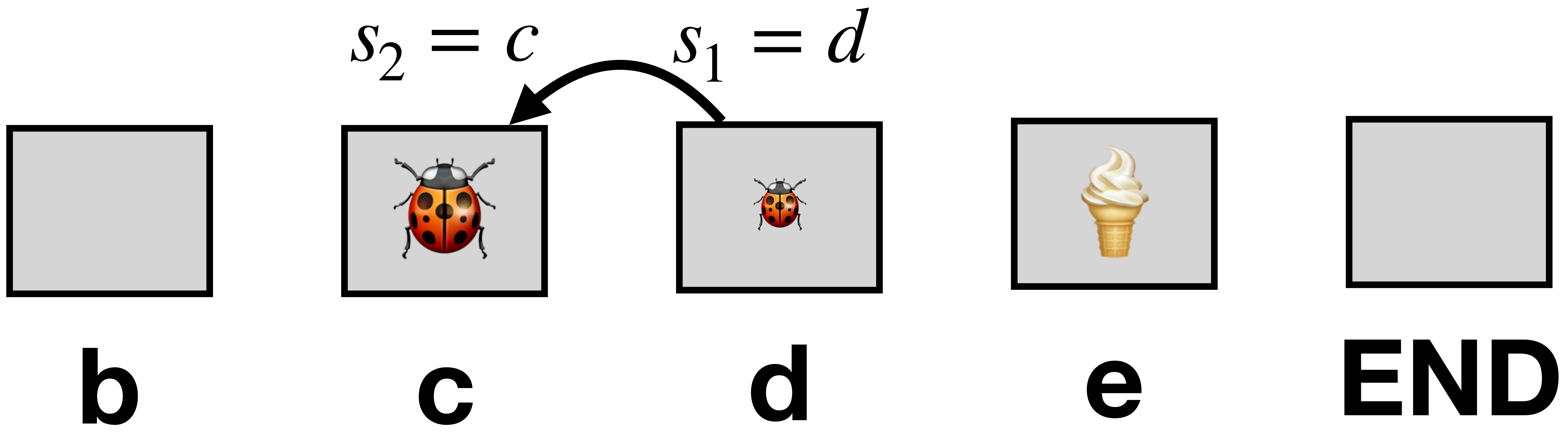
We observe: $r_1 = 1$ and $s_2 = c$



Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example ($t = 1$)

Let's run SARSA on our running example ($\gamma = 0.5$):



We will use $\varepsilon_t = 1/t$.

$t = 1,\ \varepsilon = 1,\ \alpha = 0.1$

We have: $r_1 = 1$ **and** $s_2 = c$

We sample a_2 (we are not taking it yet)

$$a_2 \sim \pi_1(a \mid c) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Say, it is $a_2 = \text{left}$.

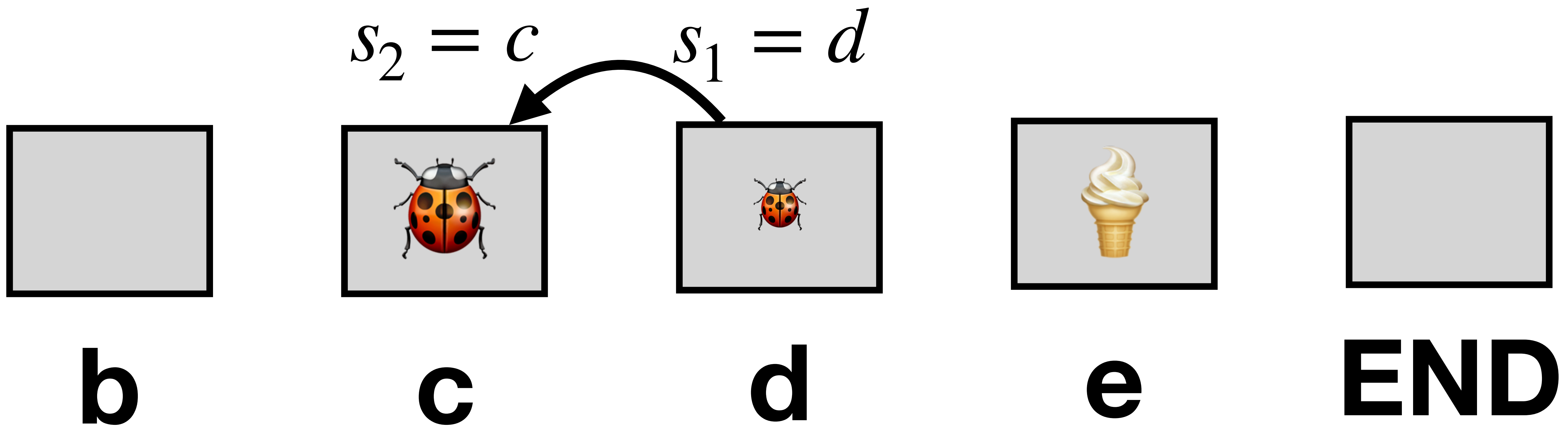
Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example ($t = 1$)

Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.

$t = 1, \varepsilon = 1, \alpha = 0.1$



We have: $r_1 = 1$ **and** $s_2 = c$

We sample a_2 (we are not taking it yet)

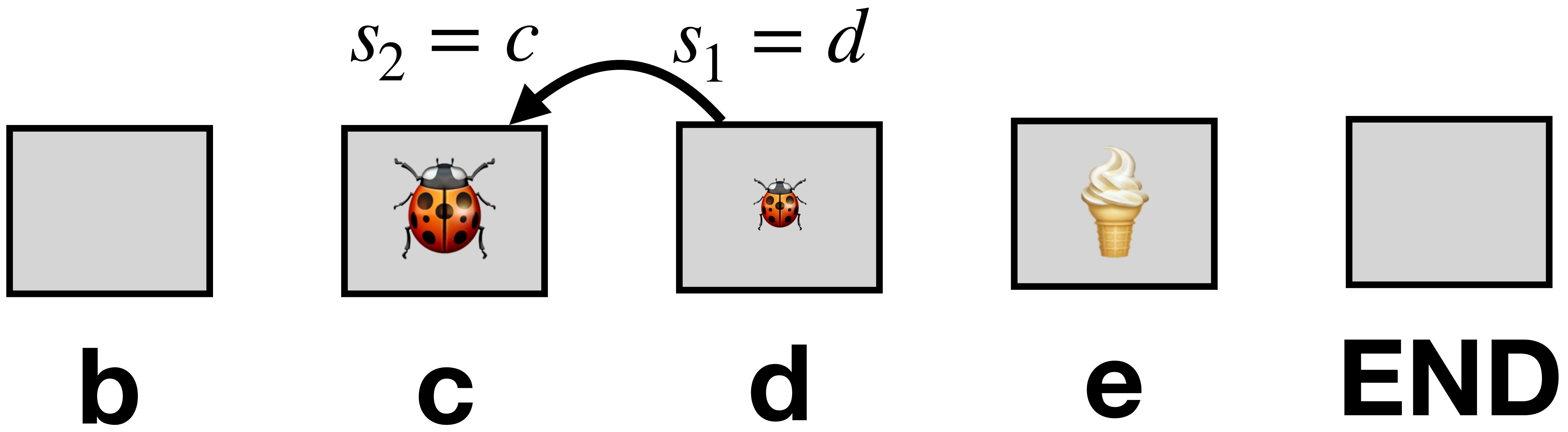
$$a_2 \sim \pi_1(a \mid c) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Say, it is $a_2 = \text{left}$.

Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example ($t = 1$)

Let's run SARSA on our running example ($\gamma = 0.5$):



We will use $\varepsilon_t = 1/t$.

$t = 1$, $\varepsilon = 1$, $\alpha = 0.1$

We have: $r_1 = 1$ **and** $s_2 = c$

We sample a_2 (we are not taking it yet)

$$a_2 \sim \pi_1(a \mid c) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Say, it is $a_2 = \text{left}$.

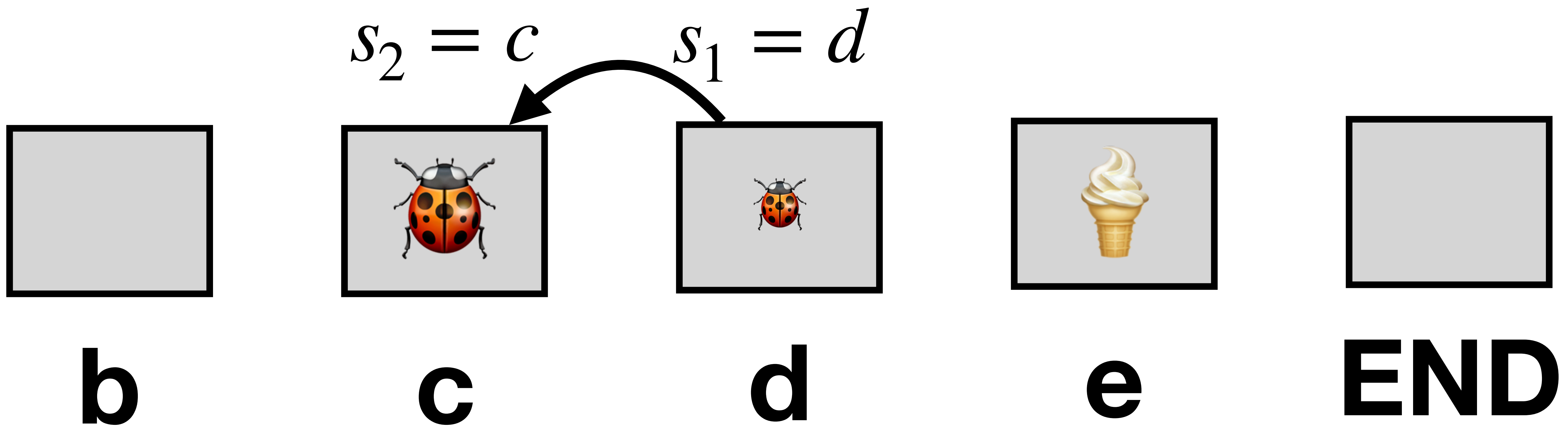
Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example ($t = 1$)

Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.

$t = 1, \varepsilon = 1, \alpha = 0.1$



We have: $r_1 = 1$ **and** $s_2 = c$

We now update the Q-function:

$$Q(d, \text{left}) := 0 + 0.1\left(1 + 0.5 \cdot 0 - 0\right) = 0.1$$

$$Q(s_t, a_t) := Q(s_t, a_t) + \alpha\left(r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t)\right)$$

Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	**0.1**	0	0
e	0	0	0

Running Example ($t = 2$)

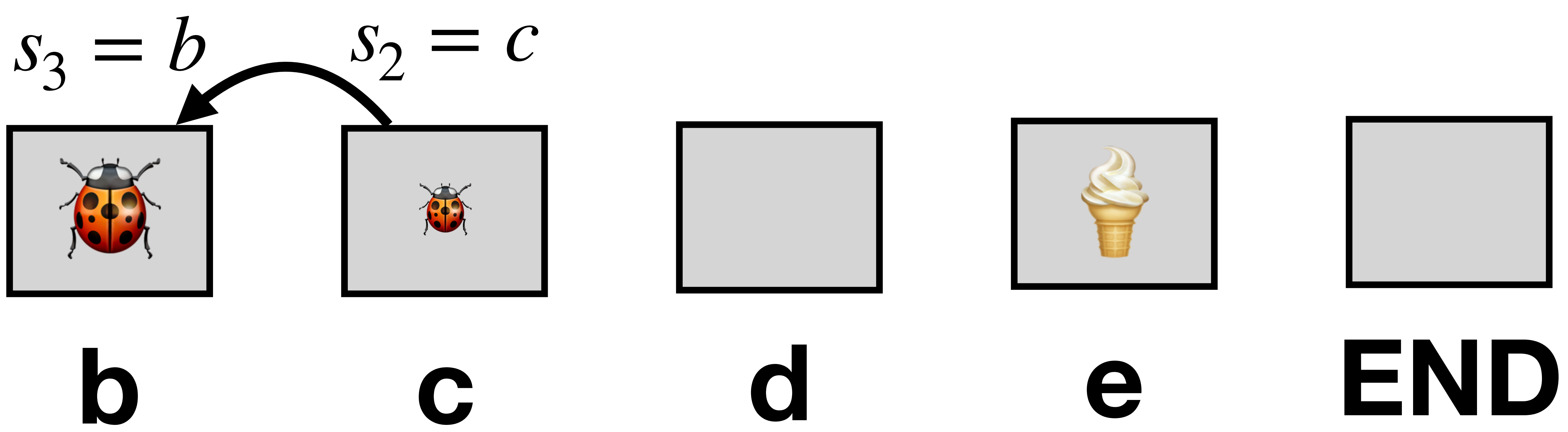
Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.

$t = 2$, $\varepsilon = 0.5$, $\alpha = 0.1$

We take the action $a_2 =$ left

We observe: $r_2 = 1$ and $s_3 = b$

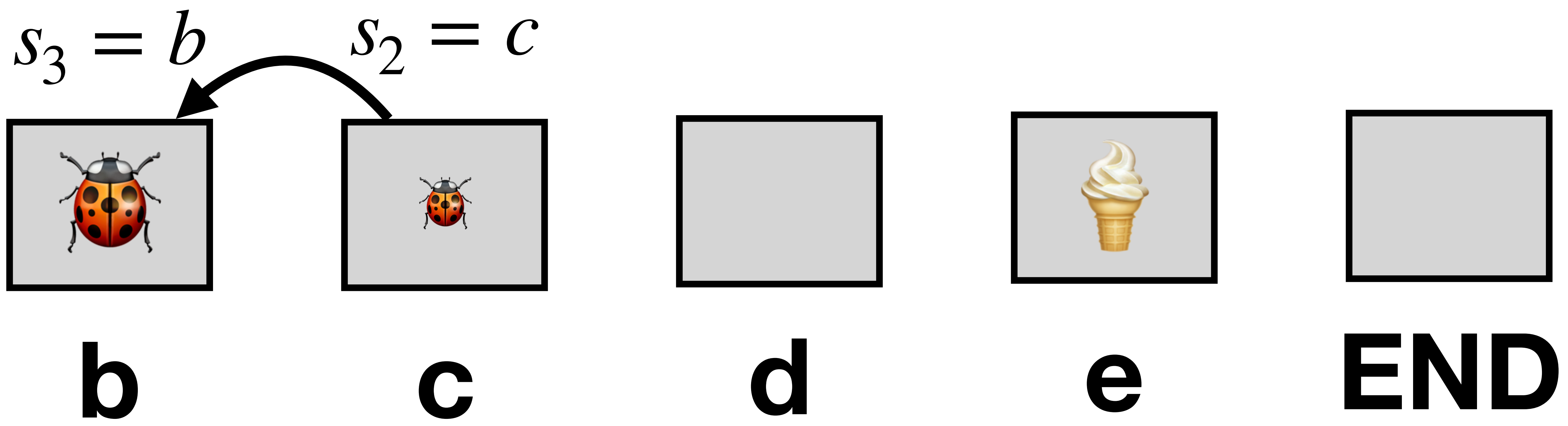


Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0.1	0	0
e	0	0	0

Running Example ($t = 2$)

Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.
$t = 2$, $\varepsilon = 0.5$, $\alpha = 0.1$



We take the action $a_2 =$ left

We observe: $r_2 = 1$ and $s_3 = b$

Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0	0	0
d	0.1	0	0
e	0	0	0

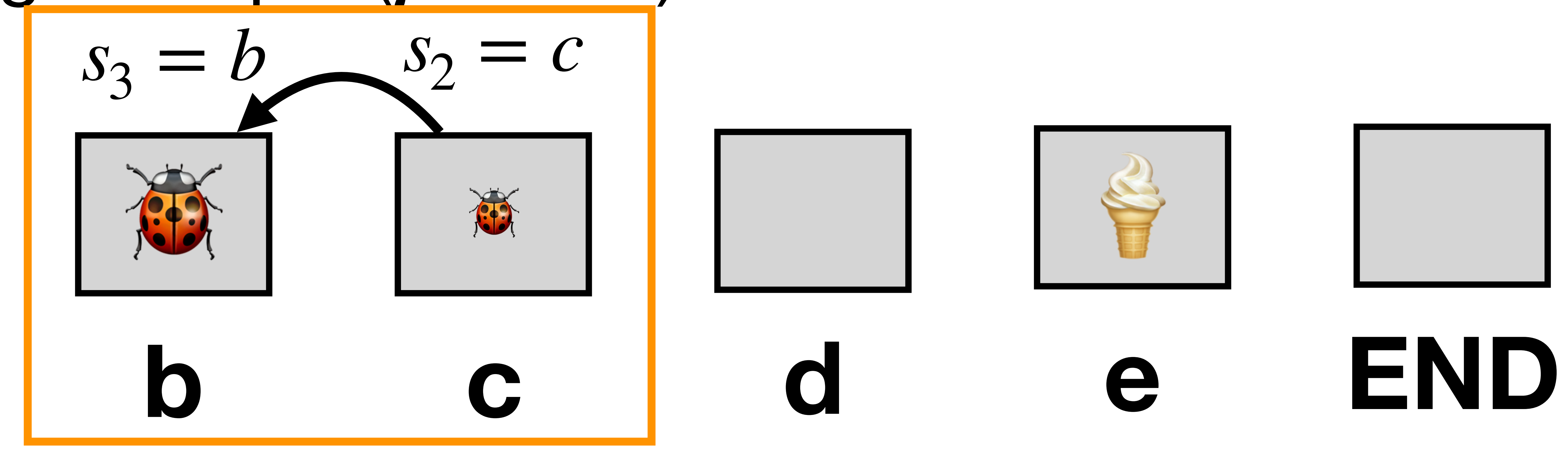
Running Example ($t = 2$)

Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.
$t = 2$, $\varepsilon = 0.5$, $\alpha = 0.1$

We take the action a_2 = left

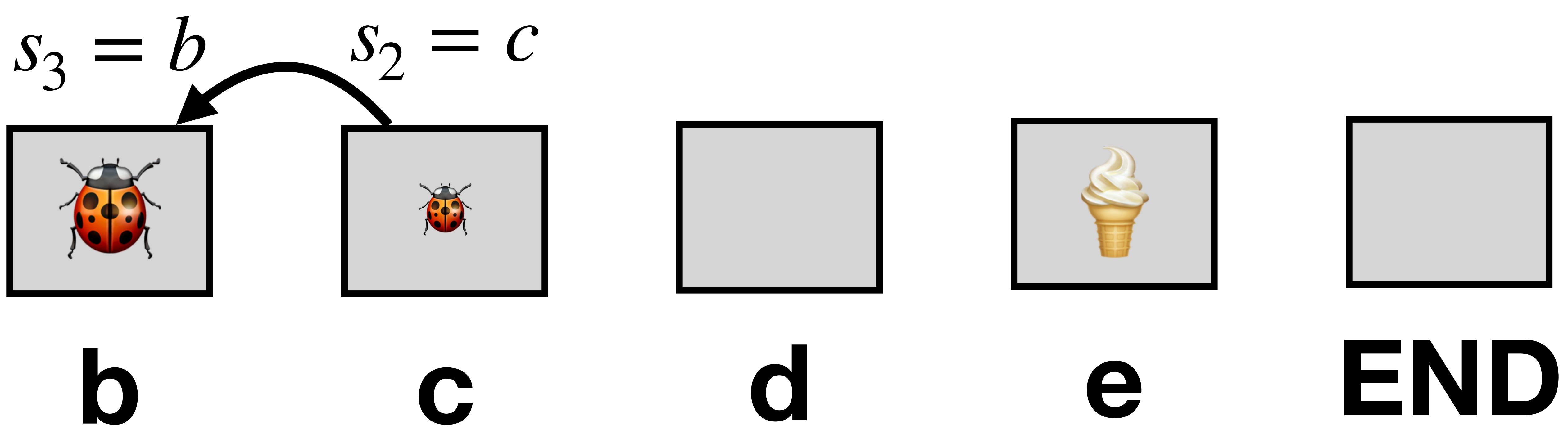
We observe: $r_2 = 1$ and $s_3 = b$



Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0.1	0	0
e	0	0	0

Running Example ($t = 2$)

Let's run SARSA on our running example ($\gamma = 0.5$):



We will use $\varepsilon_t = 1/t$.
$t = 2$, $\varepsilon = 0.5$, $\alpha = 0.1$

We have: $r_2 = 1$ **and** $s_3 = b$

We sample a_3 (we are not taking it yet)

$$\pi_1(a \mid b) = \begin{cases} 1 - 0.5 + 1/6 = 2/3 & a = \text{left} \\ 1/6 & a = \text{right} \\ 1/6 & a = \text{eat} \end{cases}$$

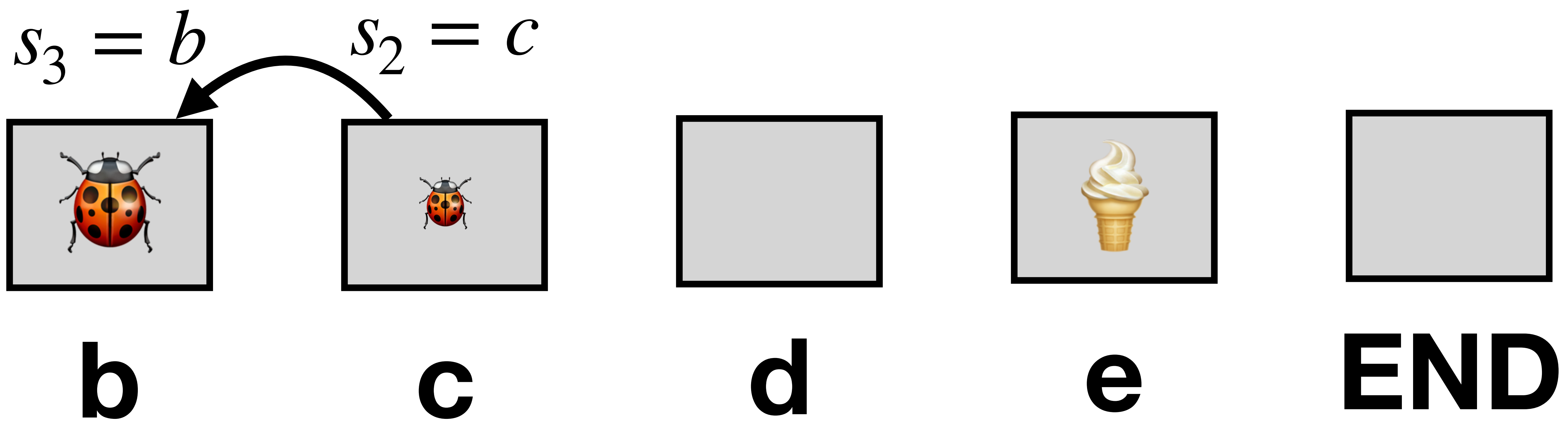
Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0.1	0	0
e	0	0	0

What happened here: Even though we did not update the estimates of the Q-function for the state c, the policy changed. Recall that we break ties (we have the preference **eat** $\prec$ **right** $\prec$ **left** and recall how we define greedy and ε-greedy policies.

Say, it is $a_3 = \text{right}$.

Running Example ($t = 2$)

Let's run SARSA on our running example ($\gamma = 0.5$):



We will use $\varepsilon_t = 1/t$.
$t = 2,\ \varepsilon = 0.5,\ \alpha = 0.1$

We have: $r_2 = 1$ **and** $s_3 = b$

We sample a_3 (we are not taking it yet)

$$\pi_1(a \mid b) = \begin{cases} 1 - 0.5 + 1/6 = 2/3 & a = \text{left} \\ 1/6 & a = \text{right} \\ 1/6 & a = \text{eat} \end{cases}$$

What happened here: Even though we did not update the estimates of the Q-function for the state c, the policy changed. Recall that we break ties (we have the preference **eat** $\prec$ **right** $\prec$ **left** and recall how we define greedy and ε-greedy policies.

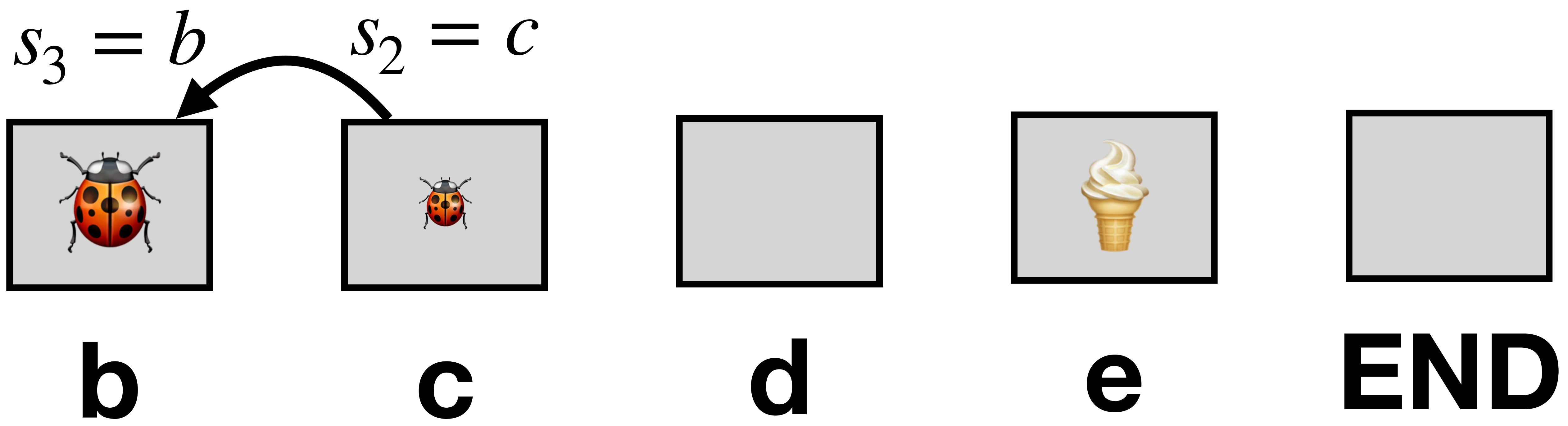
Say, it is $a_3 = \text{right}$.

Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0	0	0
d	0.1	0	0
e	0	0	0

Running Example ($t = 2$)

Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.
$t = 2, \varepsilon = 0.5, \alpha = 0.1$



We have: $r_2 = 1$ **and** $s_3 = b$

We sample a_3 (we are not taking it yet)

$$\pi_1(a \mid b) = \begin{cases} 1 - 0.5 + 1/6 = 2/3 & a = \text{left} \\ 1/6 & a = \text{right} \\ 1/6 & a = \text{eat} \end{cases}$$

Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0	0	0
d	0.1	0	0
e	0	0	0

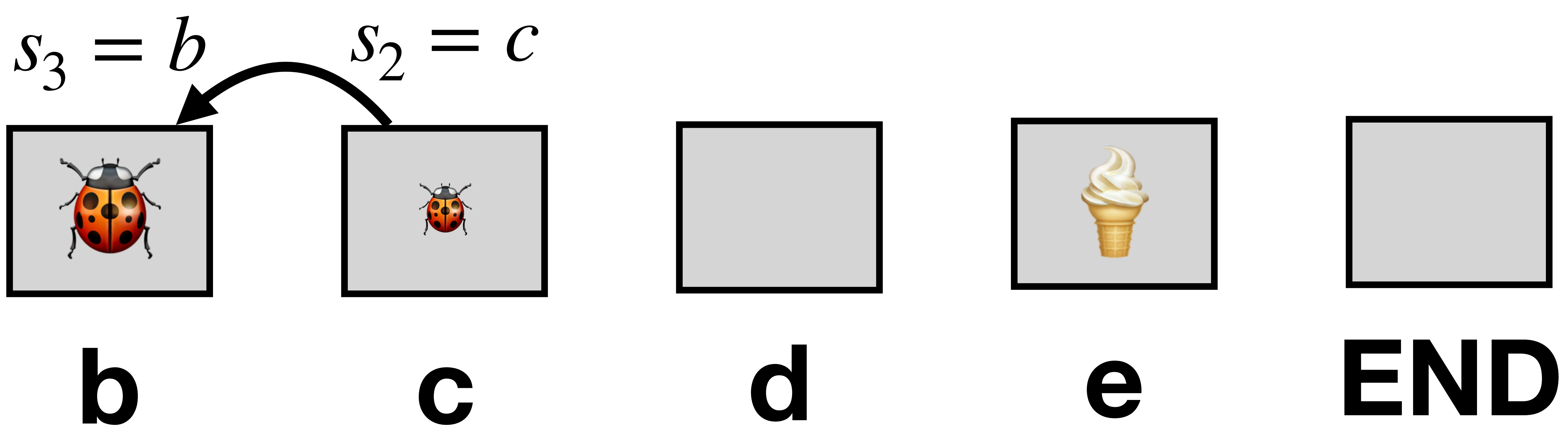
What happened here: Even though we did not update the estimates of the Q-function for the state c, the policy changed. Recall that we break ties (we have the preference **eat** $\prec$ **right** $\prec$ **left** and recall how we define greedy and ε-greedy policies.

Say, it is $a_3 = \text{right}$.

Running Example ($t = 2$)

Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.
$t = 2,\ \varepsilon = 0.5,\ \alpha = 0.1$



We have: $r_2 = 1$ **and** $s_3 = b$

We now update the Q-function:

$$Q(c, \text{left}) := 0 + 0.1\left(1 + 0.5 \cdot 0 - 0\right) = 0.1$$

$$Q(s_t, a_t) := Q(s_t, a_t) + \alpha\left(r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t)\right)$$

Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	**0.1**	0	0
d	0.1	0	0
e	0	0	0

Running Example ($t = 3$)

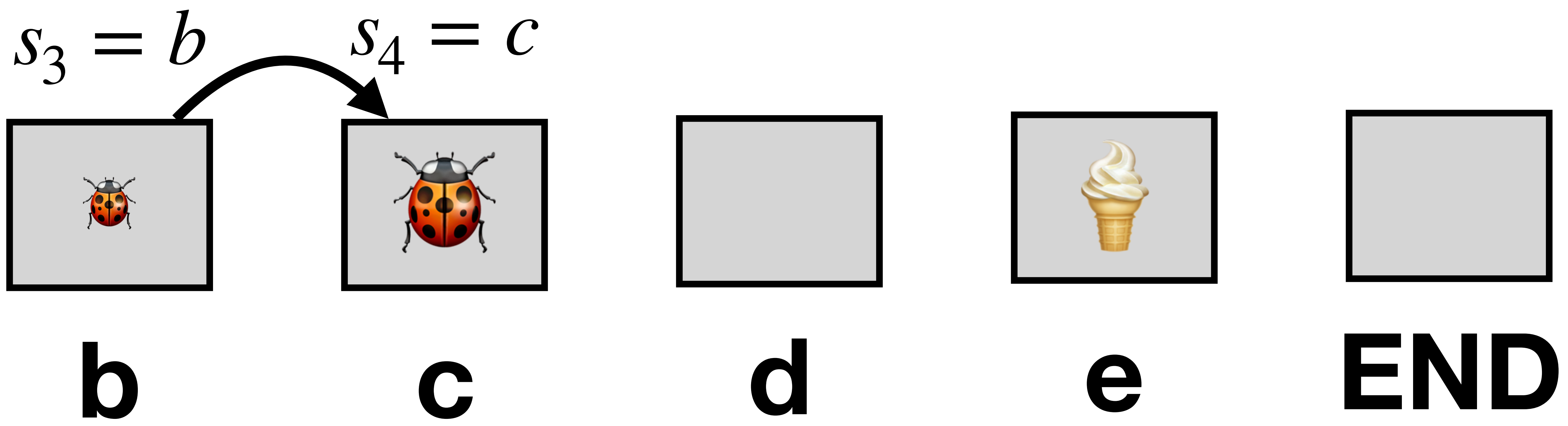
Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.

$t = 1$, $\varepsilon = 1$, $\alpha = 0.1$

We take the action $a_3 = \text{right}$

We observe: $r_3 = 1$ and $s_4 = c$.



Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	0.1	0	0
d	0.1	0	0
e	0	0	0

Running Example ($t = 3$)

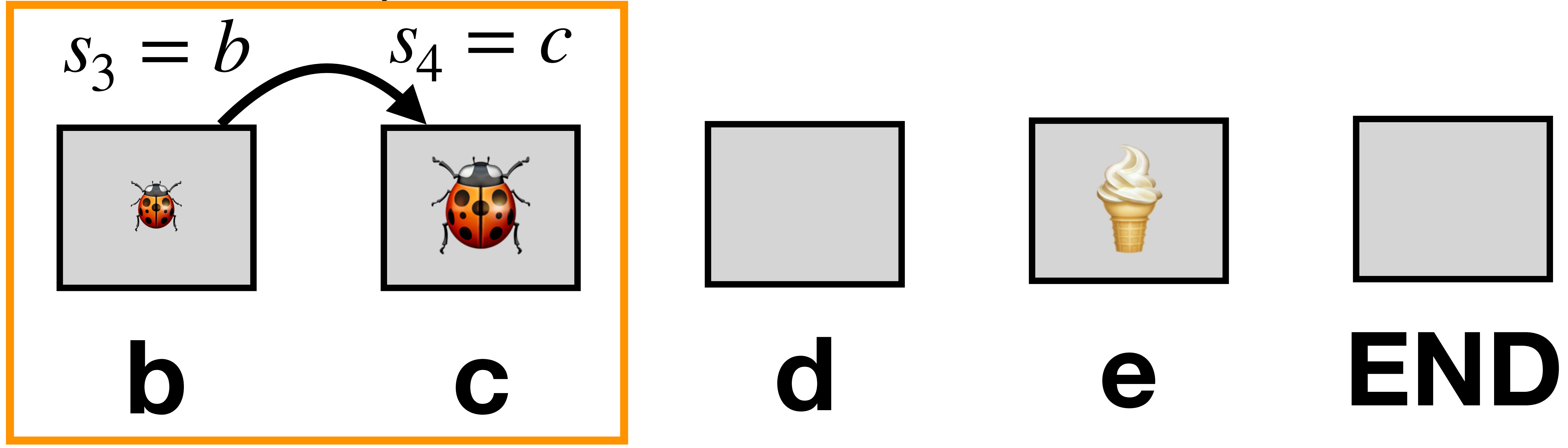
Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.

$t = 1,\ \varepsilon = 1,\ \alpha = 0.1$

We take the action a_3 = right

We observe: $r_3 = 1$ and $s_4 = c$.



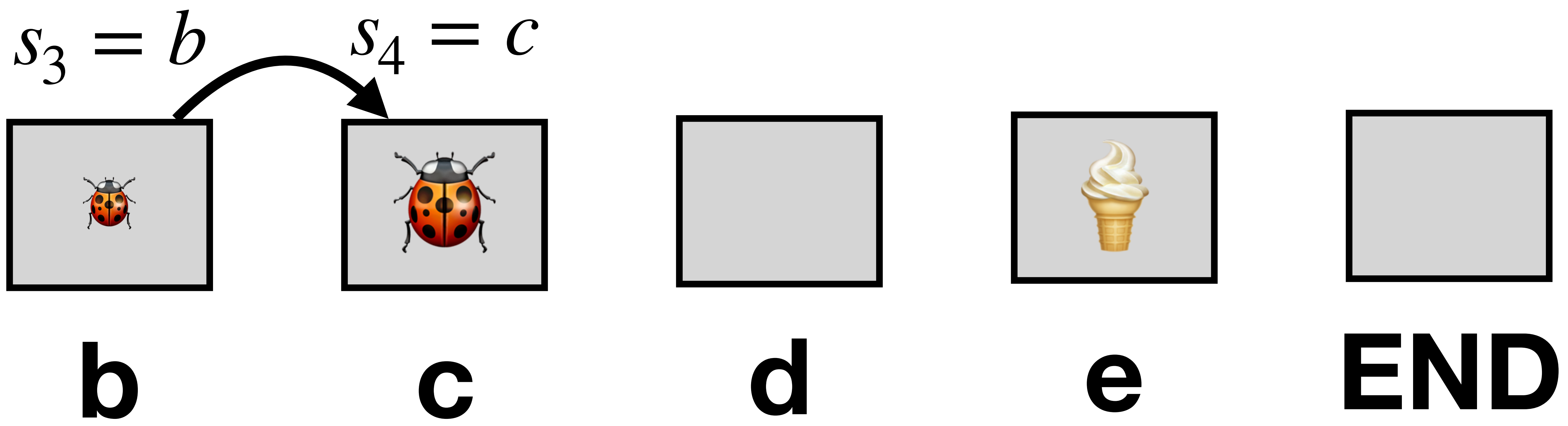
Q(s,a)	left	right	eat
b	0	0	0
c	0.1	0	0
d	0.1	0	0
e	0	0	0

Running Example ($t = 3$)

Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.

$t = 1$, $\varepsilon = 1$, $\alpha = 0.1$



We have: $r_3 = 1$ **and** $s_4 = c$

We sample a_4 (we are not taking it yet)

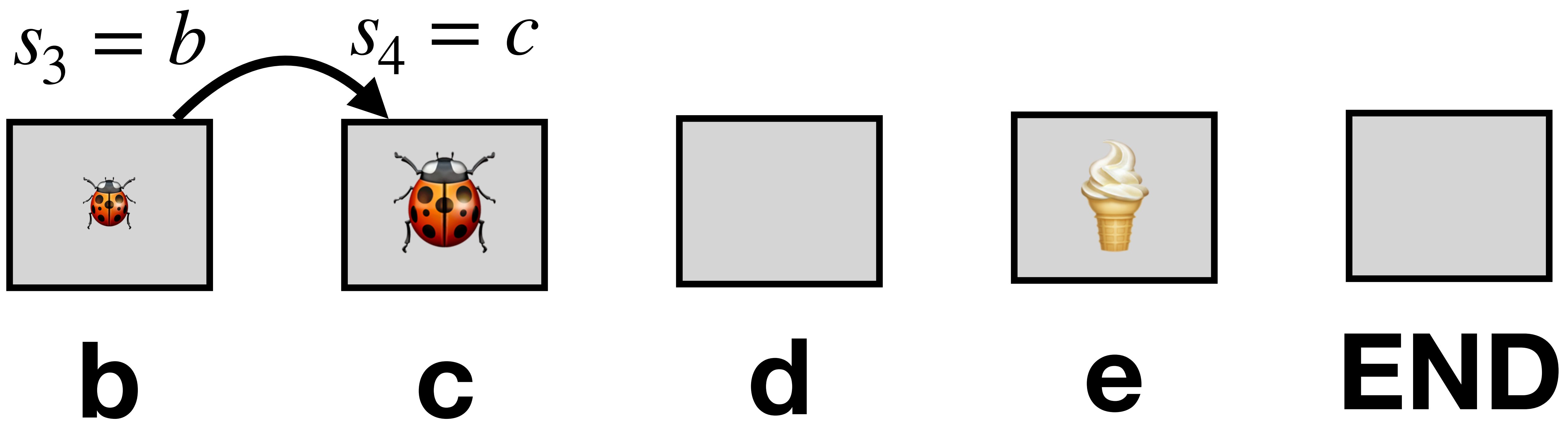
$$\pi_1(a \mid c) = \begin{cases} 7/9 & a = \text{left} \\ 1/9 & a = \text{right} \\ 1/9 & a = \text{eat} \end{cases}$$

Say, it is $a_4 =$ left.

Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0.1	0	0
d	0.1	0	0
e	0	0	0

Running Example ($t = 3$)

Let's run SARSA on our running example ($\gamma = 0.5$):



We will use $\varepsilon_t = 1/t$.

$t = 1$, $\varepsilon = 1$, $\alpha = 0.1$

We have: $r_3 = 1$ **and** $s_4 = c$

We sample a_4 (we are not taking it yet)

$$\pi_1(a \mid c) = \begin{cases} 7/9 & a = \text{left} \\ 1/9 & a = \text{right} \\ 1/9 & a = \text{eat} \end{cases}$$

Say, it is $a_4 = \text{left}$.

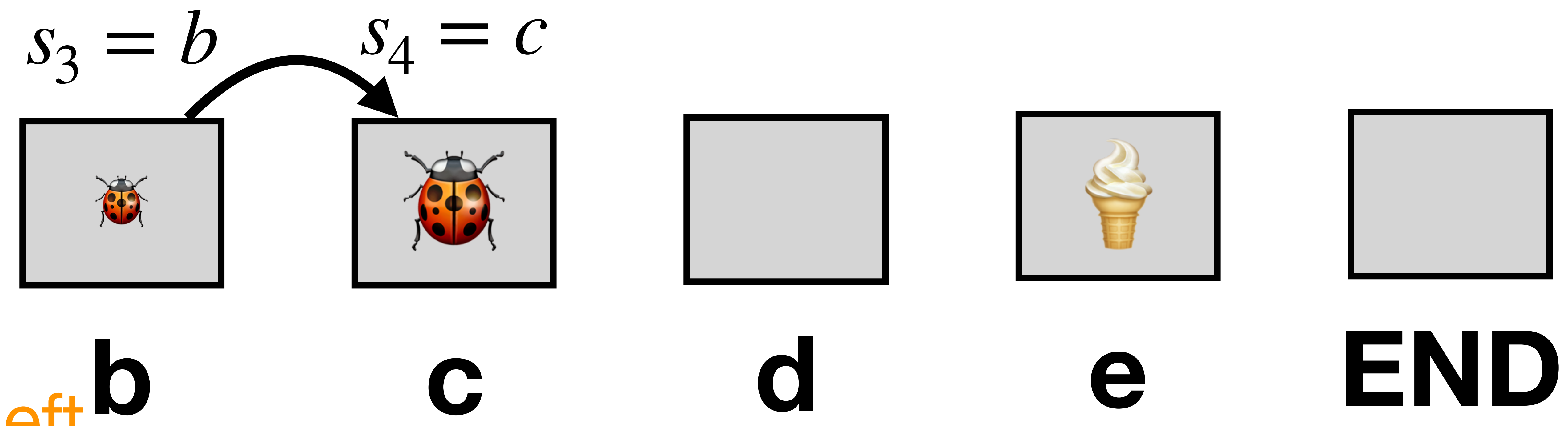
Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0.1	0	0
d	0.1	0	0
e	0	0	0

Running Example ($t = 3$)

Let's run SARSA on our running example ($\gamma = 0.5$):

We will use $\varepsilon_t = 1/t$.

$t = 1,\ \varepsilon = 1,\ \alpha = 0.1$



We have: $r_3 = 1$ **and** $s_4 = c$, $a_4 =$ left

We now update the Q-function:

$$Q(c, \text{left}) := 0.1 + 0.1 \cdot (1 + 0.5 \cdot 0.1 - 0.1) = 0.195$$

$$Q(s_t, a_t) := Q(s_t, a_t) + \alpha \left(r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t) \right)$$

Q(s,a)	*left*	*right*	*eat*
b	0	0	0
c	**0.195**	0	0
d	0.1	0	0
e	0	0	0

AND SO ON….

Note: Breaking Ties

It is usually suggested as a good idea to break ties randomly.

Indeed, as we saw in our example, without tie breaking our Q-values were prefering some actions in states we have not even visited yet, just because of the arbitrary tie breaking.

Let us rerun the example where we define the greedy policy with random tie breaking and ε-greedy policy as:

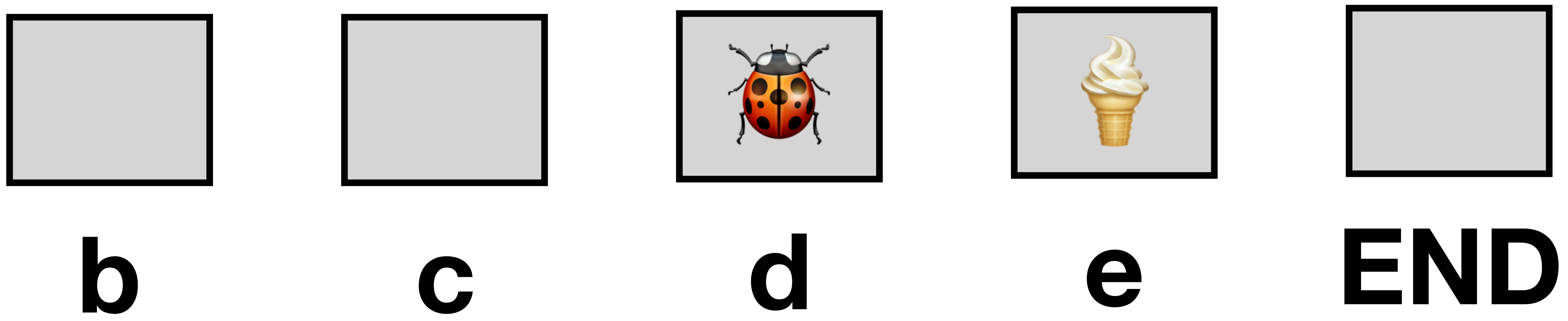
$$\pi_\varepsilon(a \mid s) = (1 - \varepsilon) \cdot \pi_{\text{greedy}}(a \mid s) + \frac{\varepsilon}{|A|}.$$

Note: We will not be showing all details of the updates in the next slides (that would be redundant to what we already saw). Focus mostly on the ε-policies.

Running Example (Initialization)

We will use $\varepsilon_t = 1/t$.

$t = 1,\ \varepsilon = 1,\ \alpha = 0.1$



With random tie-breaking:

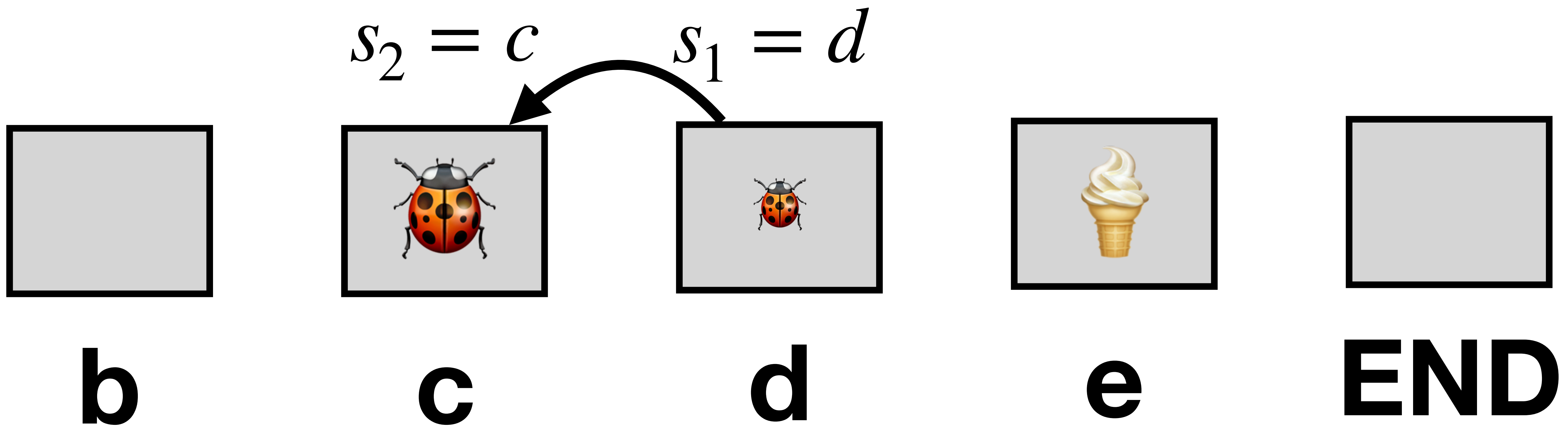
$$a_1 \sim \pi(a \mid d) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Without random tie-breaking:

$$a_1 \sim \pi(a \mid d) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example ($t = 1$)



With random tie-breaking:

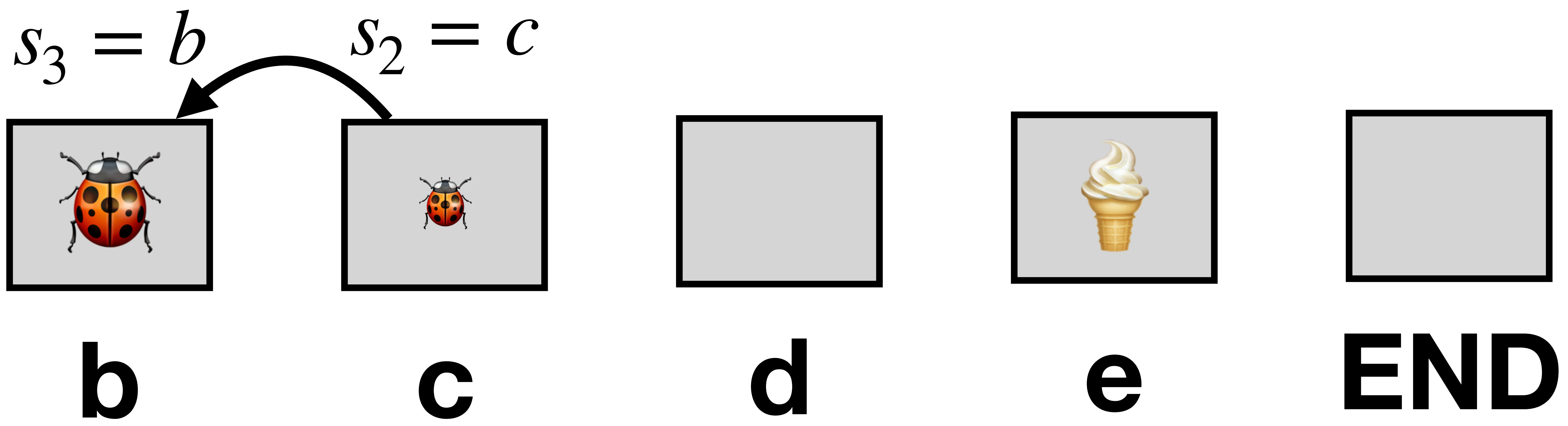
$$a_2 \sim \pi(a \mid c) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Without random tie-breaking:

$$a_2 \sim \pi(a \mid c) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0	0	0
d	0	0	0
e	0	0	0

Running Example ($t = 2$)



With random tie-breaking:

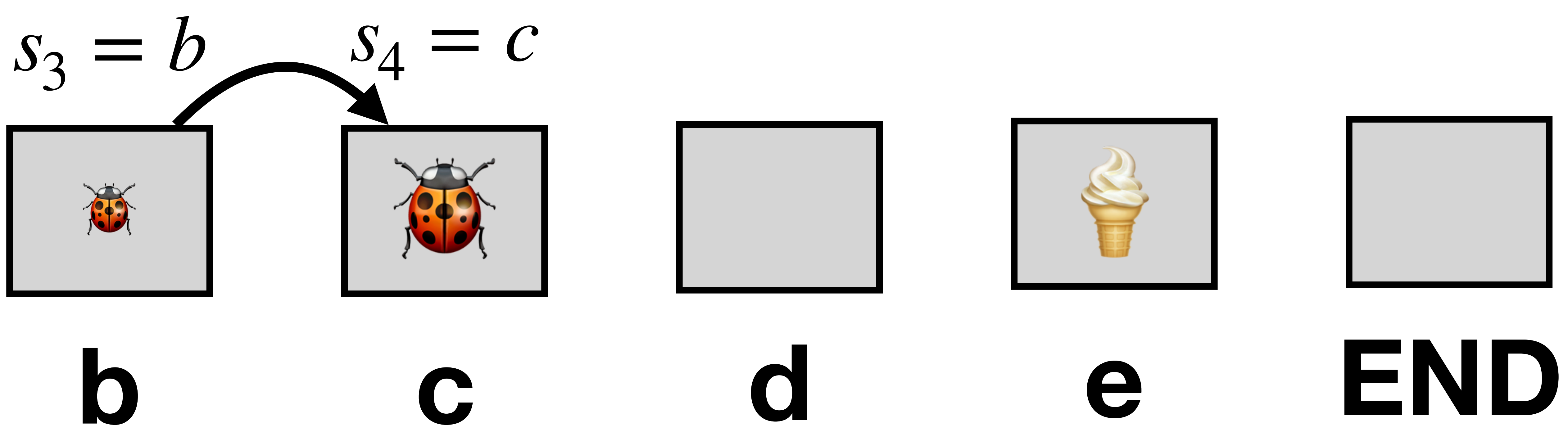
$$a_3 \sim \pi(a \mid b) = \begin{cases} 1/3 & a = \text{left} \\ 1/3 & a = \text{right} \\ 1/3 & a = \text{eat} \end{cases}$$

Without random tie-breaking:

$$a_3 \sim \pi(a \mid b) = \begin{cases} 2/3 & a = \text{left} \\ 1/6 & a = \text{right} \\ 1/6 & a = \text{eat} \end{cases}$$

Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0	0	0
d	0.1	0	0
e	0	0	0

Running Example ($t = 3$)



With random tie-breaking:

$$a_4 \sim \pi(a \,|\, c) = \begin{cases} 7/9 & a = \text{left} \\ 1/9 & a = \text{right} \\ 1/9 & a = \text{eat} \end{cases}$$

Without random tie-breaking:

$$a_4 \sim \pi(a \,|\, c) = \begin{cases} 7/9 & a = \text{left} \\ 1/9 & a = \text{right} \\ 1/9 & a = \text{eat} \end{cases}$$

Q(s,a)	***left***	***right***	***eat***
b	0	0	0
c	0.1	0	0
d	0.1	0	0
e	0	0	0

AND SO ON….

Note: Optimistic Initialization

What happens if we initialize the Q values differently?

For instance, what would happen if we started with:

Q(s,a)	*left*	*right*	*eat*
b	5	5	5
c	5	5	5
d	5	5	5
e	5	5	5

Answer: The agent would be “exploring” more than with the initialization we used.

This is a general property. If you want to promote exploration, initialize higher estimate of the Q function.

Convergence (SARSA)

- SARSA converges to the optimal state-value function Q^* if the following conditions are satisfied:

    1. The sequence of policies π_t satisfies the GLIE conditions (enough to have $\varepsilon_t = 1/t$).

    2. Step-sizes satisfy the Robbins-Monro conditions:

$$\sum_{t=1}^{\infty} \alpha_t = \infty,$$

$$\sum_{t=1}^{\infty} \alpha_t^2 < \infty.$$

Note: Why “SARSA”?

Why the name? Because of the update rule

$$Q(s_t, a_t) := Q(s_t, a_t) + \alpha \left(r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t) \right)$$

which uses the tuple $s_t, a_t, r_t, s_{t+1}, a_{t+1}$ ~ s a r s a.

Q-Learning (1/2)

- The **Optimal Bellman Equation** (we have not talked about it yet but it is similar to what we already saw)**:**

$$Q^*(s,a) = R(s,a) + \gamma \boxed{\sum_{s_{t+1}\in S} P(s_{t+1} \mid s_1, a_t) \cdot \max_{a_{t+1}\in A} Q^*(s_{t+1}, a_{t+1}).}$$

$$\mathbb{E}\left[\max_{a_{t+1}\in A} Q^*(X_{t+1}, a_{t+1}) \,\middle|\, X_t = s_t, A_t = a_t\right]$$

- Q-Learning update rule:

$$Q(s_t, a_t) := Q(s_t, a_t) + \alpha\left(r_t + \gamma \max_{a\in A} Q(s_{t+1}, a) - Q(s_t, a_t)\right)$$

Q-Learning (2/2)

1. **Initialize:** set π to be some ε-greedy policy, set $t = 1$
2. **Observe the initial state s_1**
3. **While** s_t is not a terminal state:
    1. **Take** action $a_t \sim \pi(s_t)$ and observe r_t, s_{t+1}.
    2. $Q(s_t, a_t) := Q(s_t, a_t) + \alpha \left(r_t + \gamma \max_{a \in A} Q(s_{t+1}, a) - Q(s_t, a_t) \right)$
    3. $\pi := \varepsilon$-greedy(Q)
    4. Set $t := t + 1$. Update ε, α /* *see next slides* */

Convergence (Q-Learning)

- For convergence of the state-value Q-function, we need only the Robbins-Monro conditions + every state-action pair needs to be visited infinitely often (with probability 1).

- For convergence of the policy to the optimal policy, we need GLIE (i.e. it needs to also be greedy in the limit…).

On-Policy and Off-Policy Methods

- **On-policy methods**: samples must be from the policy that we are learning. **Example:** SARSA, MC Policy Iteration.

- **Off-policy methods**: samples do not have to be from the policy that we are learning. **Example:** Q-Learning.

END OF SLIDES